The Semi-Weight Function Method in the Determination of Stress Intensity Factors for a Cracked Reissner's Plate

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ABSTRACT

A new method, semi–weight function method (SWF), is proposed to compute the stress intensity factor (SIF) for a Reissner's plate. The semi–weight functions independent of the integral path and boundaries are given. In particular, we obtain the expressions for the SIF K_I. We have computed the SIF for an edge–cracked Reissner's plate with four boundaries under symmetric bending, and obtained satisfactory results. This method is simple, explicit and valuable in studying engineering structures.

KEYWORDS:

crack, semi–weight function, stress intensity factor, Reissner, plate

1. INTRODUCTION

It is important to compute the stress intensity factor (SIF) for a cracked plate and shell in the safety estimation of engineering structures. Because three generalized displacements are employed in the theory of a Reissner's plate, the plate's deformation more truly is described, especially the stress–strain field near the crack tip in a cracked Reissner's plate. Thus, the theory of Reissner's plate is used extensively in studying a cracked plate (Seen in Liu (1983), Liu and Li (1984), Liu and Zhang (1988, b) et al.).

Since a sixth–order partial differential equation needs to be solved, some difficulties exist in solving for SIF of the cracked Reissner's plate. Although the Finite Element Method (FEM) can be used, it is not satisfactory for its great amount of work. We must search for a simple and explicit method to compute the SIF for a cracked Reissner's plate. In this paper, a new method, semi–weight function method (SWF), is proposed to do it.

The weight function method was first used by Bueckner in 1970. In the weight function method, the SIF can be written as

$$K_i = \int_{\Gamma_\sigma} T \cdot U^{(i)} ds + \int_\Omega F \cdot U^{(i)} d\Omega \qquad (i = \mathrm{I}, \mathrm{II}, \mathrm{III})$$

where $U^{(i)}$are the weight functions corresponding to Mode I, II and III. T is the boundary traction, F is the body force. If the weight functions are known for a cracked body, the SIF can be obtained simply from the above formula under arbitrary load. But the weight functions have to satisfy the equilibrium equations and the homogeneous boundary conditions, its determination is usually equivalent to compute SIF directly. Therefore, it's necessary to make some improvements on it so that it is possible to compute SIF easily.

Liu and Zhang (1988, a) defined semi–weight functions for plane problem, and gave the relationship between SWF and SIF. Better results were obtained. In this paper we give the semi–weight functions and the SIF formula for a cracked Reissner's plate, and consider the edge–cracked Reissner's plate with four free boundaries under symmetric bending. Satisfactory results are obtained.

2. THE SEMI–WEIGHT FUNCTIONS FOR REISSNER'S PLATE

The basic equations for Reissner's plate can be written as follows.
a). The equilibrium equations

$$\begin{cases} \Delta^2 F = P / D \\ \frac{1}{2}(1 - \nu) D \Delta f - Cf = C \cdot Re\Phi \end{cases} \qquad (2-1)$$

b). The geometric equations

$$\begin{cases} \psi_r = F_{,r} + \frac{1}{r} f_{,\theta} \\ \psi_\theta = F_{,\theta} / r - f_{,r} \\ W = F - \frac{D}{C}\Delta F + Im\Phi \end{cases} \qquad (2-2)$$

c). The constitutive equations

$$\begin{cases} M_r = -D[\psi_{r,r} + \nu(\psi_{\theta,\theta} + \psi_r)/r] \\ M_\theta = -D[(\psi_{\theta,\theta} + \psi_r)/r + \nu\psi_{r,r}] \\ M_{r\theta} = -\frac{1}{2}(1-\nu)D[\psi_{r,\theta}/r + \psi_{\theta,r} - \psi_\theta / r] \\ Q_r = C(W_{,r} - \psi_r) \\ Q_\theta = C(W_{,\theta}/r - \psi_\theta) \end{cases} \qquad (2-3)$$

d). Conditions on the crack surfaces $\theta = \pm\pi$:

$$M_\theta = 0; \quad M_{r\theta} = 0; \quad Q_\theta = 0 \tag{2-4}$$

where Δ is Laplace operator, D is the bending stiffness, C is the shear stiffness, Φ is an analytical function, and P is the uniform loads on the surface of plates.

It is very complex to solve the Eqs.(2−1)—(2−4) exactly. But it is not very difficult to find a solution only satisfying (2−1)—(2−3). Thus, semi−weight functions $F^{(s)}$, $f^{(s)}$, $\Phi^{(s)}$are suggested. They require

1). $F^{(s)}$, $f^{(s)}$, $\Phi^{(s)}$ satisfy Eq.(2−1)—(2−3).
2). Displacement $U_i(F^{(s)}, f^{(s)}, \Phi^{(s)}) = O(r^{-1/2})$.

Let

$$\begin{cases} F = r^{\frac{3}{2}}\{A \cdot cos\frac{\theta}{2} + B \cdot sin\frac{\theta}{2} + p \cdot cos\frac{3\theta}{2} + q \cdot sin\frac{3\theta}{2}\} \\ f = r^{\frac{1}{2}}\{m \cdot cos\frac{\theta}{2} + n \cdot sin\frac{\theta}{2}\} \\ Re\Phi = -r^{\frac{1}{2}}\{m \cdot cos\frac{\theta}{2} + n \cdot sin\frac{\theta}{2}\} \\ Im\Phi = -r^{\frac{1}{2}}\{m \cdot sin\frac{\theta}{2} - n \cdot cos\frac{\theta}{2}\} \end{cases} \tag{2-5}$$

Substitute them into Eq.(2−1). Eq.(2−5) satisfies Eq.(2−1) under zero load, but it can't satisfy Eq.(2−4). If we let $\theta = \pm\pi$:

$$M_\theta = 0; \quad Q_\theta = 0 \tag{2-6}$$

and consider its symmetry, we have

$$A = 0; \quad B = 0; \quad q = 0; \quad m = 0 \tag{2-7}$$

Thus, Eq.(2−5) becomes

$$\begin{cases} F = Ar^{\frac{3}{2}} \cdot cos\frac{3\theta}{2} \\ f = Br^{\frac{1}{2}} \cdot sin\frac{\theta}{2} \\ Re\Phi = -Er^{\frac{1}{2}} \cdot sin\frac{\theta}{2} \\ Im\Phi = Er^{\frac{1}{2}} \cdot cos\frac{\theta}{2} \end{cases} \tag{2-8}$$

Substituting Eq.(2−8) to Eq.(2−2) and (2−3), we can find the expressions for displacements and internal forces. The displacements have a singularity $r^{-1/2}$, and the internal forces $r^{-3/2}$. In fact Eq.(2−8) is just the semi−weight functions we are seeking for.

3. THE SIF OF REISSNER'S PLATE EXPRESSED IN SWF

Consider a rectangle plate. Its thickness is H, width W and length 2L. There is an edge crack in the middle of the plate. Its length is a. With the crack tip removed, the reciprocal theorem can be used

$$\begin{aligned}&\int_{\Omega}(m_x^{(s)}\psi_x+m_y^{(s)}\psi_y+P^{(s)}W)dxdy+\oint_C(Q_n^{(s)}W-M_n^{(s)}\psi_n-M_{ns}^{(s)}\psi_s ds\\&=\int_{\Omega}(m_x\psi_x^{(s)}+m_y\psi_y^{(s)}+PW^{(s)})dxdy+\oint_C(Q_nW^{(s)}-M_n\psi_n^{(s)}-M_{ns}\psi_s^{(s)})ds\end{aligned} \quad (3-1)$$

The two groups of displacements and loads with and without upperscript (s) are arbitrary, which satisfy the equilibrium equations. If we define the group with upperscript (s) as the semi–weight functions solved above, and the other as actual displacements and loads for the elastic body, Eq.(3–1) still stands.

If there is no load on the surface of plates in two cases, Eq.(3–1) is simplified into

$$\int_C(Q_n^{(s)}W-M_n^{(s)}\psi_n-M_{ns}^{(s)}\psi_s)ds=\int_C(Q_nW^{(s)}-M_n\psi_n^{(s)}-M_{ns}\psi_s^{(s)})ds \quad (3-2)$$

Let the surrounding boundary $C=C_s+C_e+\Gamma$, where C_e is the crack tip boundary after the crack tip is removed, C_s is the crack surface, Γ is the elastic body's external boundary excluding the crack boundary. If we take into account the fact that on the crack surface $\theta=\pm\pi$:

$$M_n=0;\ M_{ns}=0;\ Q_n=0 \quad (3-3)$$

Eq.(3–2) can be transformed into

$$\begin{aligned}&\int_{C_e}[(Q_nW^{(s)}-M_n\psi_n^{(s)}-M_{ns}\psi_s^{(s)})-(Q_n^{(s)}W-M_n^{(s)}\psi_n-M_{ns}^{(s)}\psi_s)]ds\\&=\int_{\Gamma}[(Q_n^{(s)}W-M_n^{(s)}\psi_n-M_{ns}^{(s)}\psi_s)-(Q_nW^{(s)}-M_n\psi_n^{(s)}-M_{ns}\psi_s^{(s)})]ds\\&\qquad+\int_{C_s}(Q_n^{(s)}W-M_n^{(s)}\psi_n-M_{ns}^{(s)}\psi_s)ds\end{aligned} \quad (3-4)$$

Near the crack tip ($r\to 0$), we have

$$Q_n \backsim r^{-\frac{1}{2}}K_IQ_n'(\theta);\ M_n \backsim r^{-\frac{1}{2}}K_IM_n'(\theta);\ M_{ns} \backsim r^{-\frac{1}{2}}K_IM_{ns}'(\theta) \quad (3-5)$$

and

$$\psi_n \backsim r^{\frac{1}{2}}K_I\psi_n'(\theta);\ \psi_s \backsim r^{\frac{1}{2}}K_I\psi_s'(\theta);\ W \backsim r^{\frac{1}{2}}K_IW'(\theta) \quad (3-6)$$

Let

$$\begin{aligned}\frac{1}{K_o}=&\int_{-\pi}^{+\pi}[(Q_n'W^{(s)}-M_n'\psi_n^{(s)}-M_{ns}'\psi_s^{(s)})r^{-\frac{1}{2}}-\\&-(Q_n^{(s)}W'-M_n^{(s)}\psi_n'-M_{ns}^{(s)}\psi_s')r^{\frac{1}{2}}]rds\end{aligned} \quad (3-7)$$

then

$$K_I = K_o\{\int_\Gamma[(Q_n^{(s)}W - M_n^{(s)}\psi_n - M_{ns}^{(s)}\psi_s) - (Q_nW^{(s)} - M_n\psi_n^{(s)} - M_{ns}\psi_s^{(s)})]ds$$
$$+\int_{C_e}(Q_n^{(s)}W - M_n^{(s)}\psi_n - M_{ns}^{(s)}\psi_s)ds\} \qquad (3-8)$$

This is the expression for SIF. Only if the displacements and loads on the elastic body's external boundary Γ are known, can we obtain K_I from Eq.(3–8). Moreover, we know that the integral path in Eq.(3–8) can be chosen wantonly as that surrounding the crack tip.

4. COMPUTATION OF SIF

Consider a Reissner's plate with four free boundaries under symmetric bending. Its thickness is H, width W and length 2L. In the middle of it, there is an edge crack. The crack's length is a. We can calculate its SIF with SWF.

We use the expansions deduced in Liu (1983) as the actual displacements and loads. The coefficients can be determined by the boundary condition. In this paper we obtain them approximately on three points of boundaries. Results are shown in tables 1 and 2.

TABLE 1. K_I changes with the plate's width

h^*	0.5			1.0		
a^*	K_I	K_{I0}	error(%)	K_I	K_{I0}	error(%)
0.1	31.28	31.17	0.3	7.82	8.07	3.1
0.2	29.24	31.40	6.9	7.31	8.08	9.5
0.3	30.10	31.77	5.3	7.52	8.19	8.1
0.4	32.57	32.84	0.8	8.15	8.40	3.0

TABLE 2. K_I changes with the plate's thickness

a^*	0.1			0.5		
h^*	K_I	K_{I0}	error(%)	K_I	K_{I0}	error(%)
0.5	31.28	31.17	0.3	32.36	33.49	3.4
1.0	7.82	8.07	3.1	8.11	8.45	4.0
1.5	3.47	3.67	5.3	3.62	3.88	6.6
2.0	1.95	2.10	6.9	2.05	2.19	6.2

NOTE:

1. $a = 1cm$, $a^* = a/L$, $h^* = h/a$, K_I——$Kg/cm^{3/2}$

2. K_{I0} is the results of Liu and Zhang (1988, b).

From tables 1 and 2 we know that, compared with the results of Liu and Zhang (1988, b), the errors are within 5 percent in general and the maximum error is less than 10 percent. It illustrates that it is simple and time–saving to compute SIF of an edge–cracked Reissner's plate with SWF and the satisfactory results are found.

When the crack is short ($a/L<0.1$), better results are also obtained if we choose the solutions of the plate without a crack as the actual approximate solutions. The results are shown in table 3.

TABLE 3. K_I with no–crack actual displacements and loads($a^*=0.1$)

h^*	0.5	1.0	1.5	2.0	2.5
K_I	33.51	8.38	3.72	2.09	1.34
K_{I0}	31.17	8.07	3.67	2.10	1.36
error(%)	7.5	3.8	1.5	0.3	1.4

5. CONCLUSIONS

a). By analyzing the Reissner's plate with a crack, we obtain the expression of Mode–I SIF under symmetric bending and the semi–weight functions to compute K_I. The semi–weight functions are independent of loads and boundaries.

b). We have computed the SIF for an edge–cracked Reissner's plate with four free boundaries under symmetric bending by the SWF. Compared with Ref.[3], the errors of the results are within 5 percent in general and the maximum is less 10 percent. Especially, if we use the solutions for the plate without a crack as the actual displacements and loads, good results are also obtained in case where the crack is short ($a/L<0.1$).

c). The computation is done on a computer 286. All results can be obtained in a few seconds. It is much faster than the numerical method FEM.

d). The SWF is simple to compute and highly–explicit. It is valuable for the study of engineering structures. It can be used in studying cracked shells, too.

6. REFERENCES

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