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An analytical solution for a transient temperature field during laser heating a finite slab

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ABSTRACT

This paper presents a general solution for the transient temperature field in a finite slab, when heated by a moving laser heat source. The analytical solution was solved by integral transform method. The eigenvalues were obtained by the method of separation of variables. In the presented model, finite region and general convective boundary conditions were assumed. Meanwhile, experiment and FEM simulation were performed as a verification under a specific case. For the experiment, a continuous wave YAG laser system was used to scan an aluminum sheet, and temperature histories at various points were logged for comparison. For the FEM simulation, a 3D transient heat transfer model was adopted with a subroutine to implement the moving laser beam. The results showed that the analytical solution was consistent with both the experimental data and FEM simulation.

1. Introduction

Lasers are widely adopted in material processing because of their highly concentrated energy distribution. There are many applications of lasers in sheet metal forming such as laser forming and laser assisted pre-stress forming. Laser forming is a flexible technique which forms sheet metal by means of laser induced non-uniform temperature field and thermal stress [1]. During the forming process, deformation is induced in a controllable manner by planning the laser scanning path in advance. It is characterized as a non-contact and non-wearing treatment. On the other hand, in laser-assisted pre-stress forming, forming is achieved by radiating strain concentrated areas with lasers, where the strain is induced by a pre-loaded stress [2]. Due to the softening effects in the heated region, material yield strength is locally reduced and the part is formed effectively. In all these forming methods, there are many factors governing the final formed shape, and even the forming mechanisms, such as laser parameters, part geometry dimensions, and boundary constraints. It was reported to be viable to predict forming ratio or bending angle in those two major laser forming techniques directly by adopting empirical modeling, without solving the temperature field. For example, Cheng and Yao [3] developed a method for laser forming of a class of shapes with genetic algorithm. Springback ratio can be predicted and controlled by modeling with neural network and large set of experimental data [4]. Pulsed laser bending of sheet metal can be modelled and optimized with neural networks and neuro-fuzzy system [5]. However, since deformation is driven by thermal field, temperature field would give a better understanding in the forming mechanisms. On the other hand, temperature field is of great importance in procedure planning, which is a dominant key for desired or undesired material transition. For instance, there is a maximal permissible temperature value to ensure that the workpiece is not compromised during

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forming process [2]. For such purposes, efforts have been made to solve the temperature field via various approaches such as numerical simulations, artificial intelligence techniques and analytical methods.

Numerical simulations, finite element method (FEM) or finite difference method (FDM) in most of the cases, are effective to analyze the temperature profile with given specific laser processing parameters. Ji and Wu [6] established a simplified mathematical model for the laser heating of sheet metal, analyzed the heating process with a FEM program, and compared with a FDM model. Chen et al. [7] proposed a numerical model, which combined equations of FEM with those of FDM, for the simulation of the temperature filed. Jamil et al. [8] investigated influences of different laser beam geometries on temperature profiles with FEM simulation. Yilbas and Akhtar [9] studied temperature history under multiple laser scans by introducing a volume flux in ABAQUS. With proper simplifications and assumptions, FEM simulation with a fine mesh can approximate the real-world situation. However, FEM approach is commonly much time-consuming, given that the computing cost increases with a finer mesh [10].

Recently, researchers have employed various artificial intelligence techniques for the prediction and optimization of various laser advanced machining process. Artificial Neural Network (ANN) has been widely used for modeling because of its nonlinear, adaptive, and learning properties. Ismail et al. [11] used an ANN model to establish an intelligent algorithm to build a simplified relationship between laser parameters and weld bead geometry for laser microwelding of thin steel sheet. Mishra and Yadava [12,13] proposed an ANN model, which was trained with verified FEM model and simulated data, to predict the hole taper and the material removal rates as well as the extent of HAZ during laser beam percussion drilling of both thin aluminum sheet and nickel-based super-alloy sheet. ANN is widely established in the artificial intelligence research where the output variables behavior non-linearly with the input variables. To approximate such non-linear relationship, a large set of experimental or simulation data is commonly required, especially in the area of advanced laser machining.

In sum, both ANN and FEM can be utilized to predict the real-world situation. On the other hand, analytical method provides a reasonably simplified model of abstraction by directly solving the governing differential equation for heat transfer. It is worthy to mention that ANN model, accompanied with an analytical model of a one dimensional heat transfer equation, does predict the achieved hardness in laser hardening of K340 steel [14].

To model the temperature field during laser forming analytically, Geiger and Vollertsen proposed a two-layer model, which presented a temperature discontinuity in the middle [1]. This model could fit the experimental data only on the very specific conditions. Vollertsen and Rodle proposed another model taking account of temperature distribution along thickness [15]. Lambiase [16] also established another two-layer model whereas the heated layer thickness depended on the effective temperature distribution along the sheet thickness. However, a stationary source was assumed and distributions along the length and the width direction were neglected. In some cases, one-dimensional model is suitable for modeling the temperature distribution along depth direction. Nath et al. [17] used such a model to investigate the temperature variations during heating and cooling cycles in laser surface hardening. Lambiase and Di Ilio [18] predicted the bending angle during laser forming by dividing and averaging accordingly the temperature field along depth direction into two layers, based on the 1-D thermal modeling. Cheng and Lin [19] proposed an analytical model with a moving Gaussian distribution source by solving the heat transfer equation with superposition and mirror-image method. Yet the solution assumed a semi-infinite slab with adiabatic boundary condition on both surfaces. Conde et al. [20] proposed two models depending on the modeling of laser source using the Green function method. However, the boundary conditions were also considered to be adiabatic. Majumdar and Xia [21] proposed a Green's function model for a two dimensional situation. Because of the absence of width direction, the effect of width on the temperature distribution cannot be calculated.

As a matter of fact, for many cases, adiabatic boundary conditions cannot be assumed to predict temperature field with enough precision. For example, Lambiase et al. proposed a passive water cooling method during laser forming process, which can be beneficial in bending angle and in cooling time [22]. In such case, intensive convective boundary condition should be considered. As another example, during laser assisted pre-stress forming of aluminum sheet, the maximal temperature is suggested to be no higher than 300°C; convective boundary significantly affects the temperature field [23].

In this paper, an analytical model for a transient temperature field during laser heating a finite slab was proposed by solving directly the general 3D heat transfer equation. Such solution was obtained by integral transform method, accompanied with separation of variables for solving the corresponding homogenous equation. To verify the model, a case that an aluminum workpiece was heated by laser line scanning was chosen. In addition, both experiment and FEM simulation were performed. Experiment was carried out by heating an aluminum sheet with continuous wave YAG laser, and logging the temperature history at interested test points. FEM simulation was realized with ABAQUS software. Verification indicated that the presented model was feasible under the given processing parameters.

2. Mathematical modeling

The problem of heating a finite slab with a laser beam of a specific energy distribution can be modelled as a three dimensional heat transfer equation, in a region of 3D Euclidean space, as shown in Fig. 1. The top surface is radiated with a moving laser beam with a specific power distribution. The length, width and thickness are denoted by L, W and H respectively. Laser beam radius is denoted by r_b. And P1, P2, and P3 are points where temperature histories are measured for model verification. The governing heat transfer equation and the boundary and initial condition can be formulated with:

$$\frac{1}{D}\frac{\partial T(x,y,z,t)}{\partial t} - \nabla^2 T(x,y,z,t) = \frac{1}{k}g(x,y,z,t), \quad \text{for } (x,y,z) \in B, \tag{1}$$

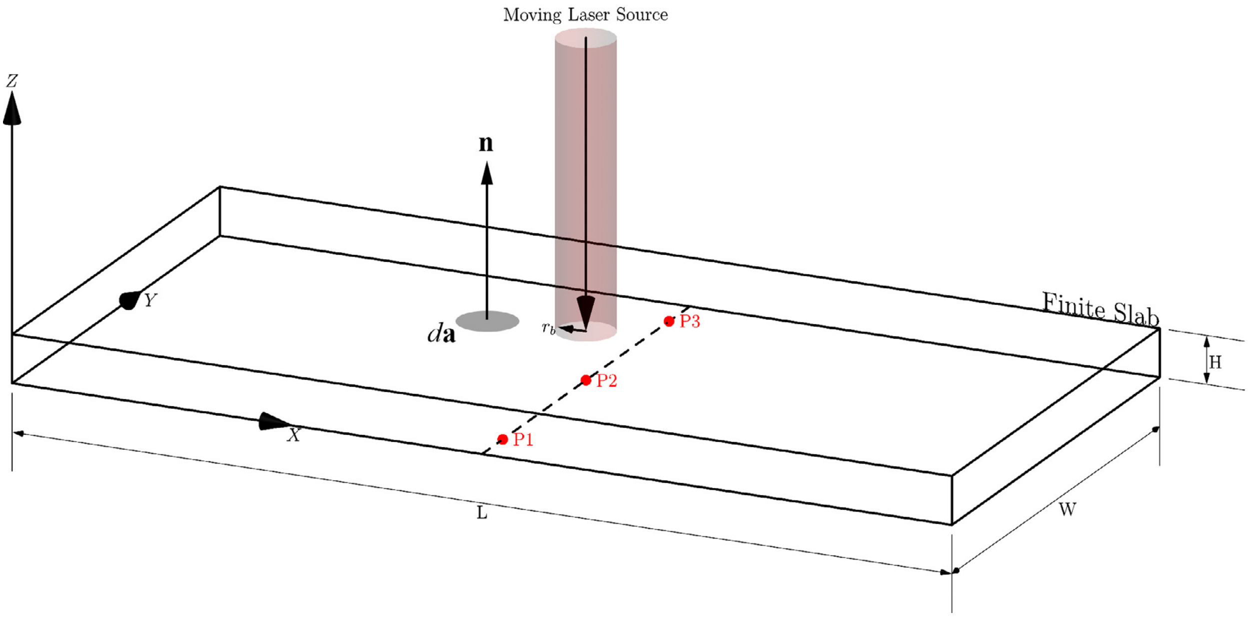

Fig. 1. Model for heating a finite slab with a moving laser source.

$$k\frac{\partial T(x,y,z,t)}{\partial \mathbf{n}} + hT(x,y,z,t) = f, \quad \text{for } (x,y,z) \in \partial B, \tag{2}$$

$$T(x,y,z,\ 0) = F(x,y,z), \quad \text{for } (x,y,z) \in B, \tag{3}$$

where D is the thermal diffusivity, k the thermal conductivity, $g(x, y, z, t)$ the heat generation rate per unit volume, $\mathbf{n}$ the outward unit normal vector of the infinitesimal element of area $d\mathbf{a}$ on the boundary surface, h the thermal convection coefficient, f the generalized heat flux along the boundary surface, F the initial temperature distribution and B is the Euclidean space configuration of the slab.

In fact, material properties are dependent with temperature, which varies nonlinearly. However, with limited range of temperature variation, material properties can be viewed as constants in order to simplify the modeling process. Therefore, it is here assumed that all material properties remain unchanged with temperature.

To solve Eq. (1), method of integral transformation can be employed [24]. First of all, one needs to solve the corresponding eigenvalue problem:

$$\nabla^2 T + \lambda^2 T = 0, \quad \text{for}(x,y,z) \in B, \tag{4}$$

$$k\frac{\partial T}{\partial \mathbf{n}} + hT = 0, \quad \text{for}(x,y,z) \in \partial B, \tag{5}$$

where λ is the eigenvalue. Second, the eigenvalues and the corresponding eigenfunctions on each dimension can be obtained by separation of variables. The eigenvalues α_l, $(l = 1, 2, 3, \dots)$ on the x direction are determined by equation:

$$\tan(\alpha_l L) = \frac{2\alpha_l \dfrac{h}{k}}{\alpha_l^2 - \left(\dfrac{h}{k}\right)^2}. \tag{6}$$

The eigenfuction associated with the eigenvalue can be written as:

$$X_l(x) = \alpha_l \cos(\alpha_l x) + \frac{h}{k}\sin(\alpha_l x), \tag{7}$$

with a normalization integral:

$$N_x(l) = \int_0^L {X_l}^2(x)dx \tag{8}$$

$$= \frac{1}{2}\left(\left(\alpha_l^2+\left(\frac{h}{k}\right)^2\right)\left(L+\frac{\frac{h}{k}}{\alpha_l^2+\left(\frac{h}{k}\right)^2}\right)+\frac{h}{k}\right). \tag{8}$$

Similarly, eigenfunctions in Y and Z directions can be written as:

$$Y_m(y) = \beta_m \cos(\beta_m y) + \frac{h}{k}\sin(\beta_m y), \tag{9}$$

$$Z_n(z) = \gamma_n \cos(\gamma_n z) + \frac{h}{k}\sin(\gamma_n z), \tag{10}$$

associated with which the corresponding normalization integrals are:

$$N_y(m) = \frac{1}{2}\left(\left(\beta_m^2+\left(\frac{h}{k}\right)^2\right)\left(W+\frac{\frac{h}{k}}{\beta_m^2+\left(\frac{h}{k}\right)^2}\right)+\frac{h}{k}\right), \tag{11}$$

$$N_z(n) = \frac{1}{2}\left(\left(\gamma_n^2+\left(\frac{h}{k}\right)^2\right)\left(H+\frac{\frac{h}{k}}{\gamma_n^2+\left(\frac{h}{k}\right)^2}\right)+\frac{h}{k}\right), \tag{12}$$

and with β_m, γ_n determined by:

$$\tan(\beta_m L) = \frac{2\beta_m \frac{h}{k}}{\beta_m^2-\left(\frac{h}{k}\right)^2}, \tag{13}$$

$$\tan(\gamma_n L) = \frac{2\gamma_n \frac{h}{k}}{\gamma_n^2-\left(\frac{h}{k}\right)^2}. \tag{14}$$

Given eigenfunctions and normalization integrals, the temperature function T can be assumed to be:

$$T(x,y,z,t) = \sum_{l,m,n} T_{lmn}(t)\frac{X_l(x)Y_m(y)Z_n(z)}{N_x(l)N_y(m)N_z(n)}. \tag{15}$$

Denoting $\Phi_{lmn} = X_l(x)Y_m(y)Z_n(z)$, with equation Eq. (1) and Green's formula, one can write:

$$\frac{1}{D}\iiint_B \Phi_{lmn}T dv = \iiint_B \Phi_{lmn}\frac{g}{k}dv + \iiint_B \Phi_{lmn}\nabla^2 T dv \tag{16}$$

$$= \iiint_B \Phi_{lmn}\frac{g}{k}dv + \iiint_B \nabla^2\Phi_{lmn}T dv + \iint_{\partial B}\left(\Phi_{lmn}\frac{\partial T}{\partial n} - T\frac{\partial \Phi_{lmn}}{\partial n}\right)ds. \tag{16}$$

With Eq. (2), (3), and (16) and the inverse transform of Eq. (15), T_{lmn} can be solved by:

$$T_{lmn} = e^{-D(\alpha_l^2+\beta_m^2+\gamma_n^2)t}\left(F_{lmn} + \int_0^t e^{D(\alpha_l^2+\beta_m^2+\gamma_n^2)\tau}A_{lmn}d\tau\right), \tag{17}$$

where

$$F_{lmn} = \iiint_B F\Phi_{lmn}dv, \tag{18}$$

$$A_{lmn} = D\iiint_B \frac{g}{k}\Phi_{lmn}dv + D\iint_{\partial B}\frac{f\Phi_{lmn}}{k}ds. \tag{19}$$

And finally, the solution to Eq. (1) can be written in the form:

$$T(x,y,z,t) = \sum_{l,m,n} e^{-D(\alpha_l^2+\beta_m^2+\gamma_n^2)t}\frac{X_l Y_m Z_n\left(F_{lmn} + \int_0^t e^{D(\alpha_l^2+\beta_m^2+\gamma_n^2)\tau}A_{lmn}d\tau\right)}{N_x(l)N_y(m)N_z(n)}. \tag{20}$$

Table. 1
Parameters used in model verification.

Parameters	Values
Sample length (L)	0.100 m
Sample width (W)	0.048 m
Sample thickness (H)	0.00254 m
Laser beam radius (r_b)	0.003 m
Laser scanning velocity (V)	0.0005 m/s
Laser output power (P)	140 W
Absorptivity	0.7
Thermal conductivity (K)	180 $\mathrm{Wm^{-1}k^{-1}}$
Specific heat (c)	1000 $\mathrm{Jkg^{-1}K^{-1}}$
Density (ρ)	2700 $\mathrm{kgm^{-3}}$
Convection coefficient (h)	60 $\mathrm{Wm^{-2}K^{-1}}$

3. Verification

The idea of verification was to prove the feasibility of the proposed model for the temperature field induced by a scanning laser beam. In order to do that, both experiment and FEM simulated were employed.

3.1. Experiment

Material used in the experiment was 6061T6 aluminum alloy sheet, of which the geometry dimensions were listed in Table 1. A continuous wave YAG laser system with a three dimensional staging system was adopted. Table 1 showed the laser processing parameters. In order to increase the energy absorption, the sample was coated with a layer of graphite ink. To verify the proposed model, temperature histories of three points were recorded whose coordinates were respectively P1(0.5L, 0.1W, 0), P2(0.5L, 0.5W, 0) and P3(0.5L, 0.9W, 0) with the coordinate system established in Fig. 1. At each point, a type K thermocouple was attached. Temperature variations during the whole machining process were logged with a HIOKI 8430-20 data logger. The sampling interval was set to be 10 ms. The experiment comprised two stages: laser heating and cooling. During heating stage, the laser moved from point (L/2, 0, H) to (L/2, W, H), and no protective gas was adopted. After that, sample was cooled naturally.

3.2. FEM simulation

In the present numerical simulation, the transient temperature field was determined based on heat transfer. Thermal transfer calculation was performed with ABAQUS, a commercial FEM code. The laser was modelled as a moving surface heat flux which was implemented with DFLUX, a user subroutine for ABAQUS. Geometry dimensions and laser parameters were list in Table 1. To simplify the analysis, three assumptions were made to solve the thermal field:

(1) The laser intensity distribution was uniform.
(2) Only convection ($h = 60\ \mathrm{W/m^2 \bullet K}$) was considered.
(3) Changes of material properties with temperature could be linearly interpolated.

For the heat transfer analysis, a symmetric model was built, in which elements of type 'DC3D8' were used. Total number of elements was 48,000. During the thermal analysis, the node temperature was calculated by solving the non-linear heat transfer equation Eq. (21).

$$[C_p(T)]\{\dot{T}(t)\} + [k(T)]\{T(t)\} = \{Q(t)\}, \tag{21}$$

where $[C_p(T)]$ and $[k(T)]$ denotes the temperature-dependent specific heat matrix and conductivity matrix, respectively, $\{Q(t)\}$ the heat generation rate per unit volume, $\{T(t)\}$ the nodal temperature vector and $\{\dot{T}(t)\}$ is the time derivative of $\{T(t)\}$. Eq. (21) was solved with a set of automatically selected time increments. During each increment, the maximal allowed temperature variation was chosen to be 20°C.

3.3. Results and verification

In the presented analytical model, generally, there are two approaches to model the heating process by a laser beam. Laser radiation can be interpreted either as a heat flux cross the boundary surface, or as a distributed volume heat source in the domain. Since the latter approach is more accurate according to the mechanism of laser material interaction, it is widely adopted [10,25–27]. Let $q(x, y, t)$ be the laser power distribution, since the absorption depth by metal is negligible, the volumetric heat source can be expressed as:

$$g(x, y, z, t) = q(x, y, t)\delta(z - H), \tag{22}$$

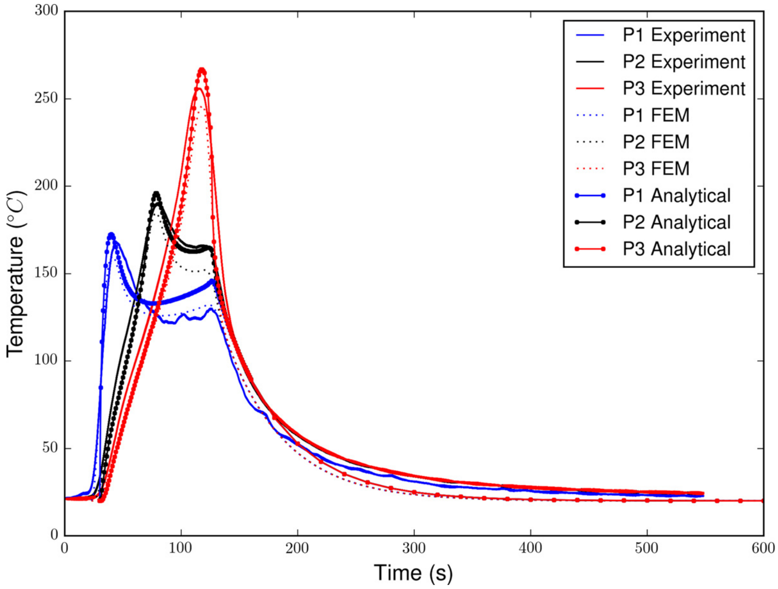

Fig. 2. Comparison between analytical solution, experimental data and FEM predictions in the temperature history at point P1, P2 and P3.

where δ is the Dirac delta function. For a top-hat distribution $q(x, y, t)$ can be expressed as:

$$q(x,y,t) = \begin{cases} \dfrac{AP}{\pi \cdot r_b^2}, & \text{if}(x,y) \in B_{r_b}(c(t)) \\ 0, & \text{if}(x,y) \notin B_{r_b}(c(t)) \end{cases}, \tag{23}$$

where A is the absorptivity, P is the laser power, r_b is the laser spot radius, and $c(t)$ is the center of laser beam at time t, and B_{r_b} is the neighborhood of $c(t)$ with a distance of r_b.

In this case, since no protective gas was used, only the natural convection is necessary to be considered. Furthermore, a uniformly distributed initial bulk temperature is assumed. These assumptions directly lead to $F_{lmn} = 0$ and $f = 0$. Finally, the temperature field can be solved with Eq. (24).

$$\begin{aligned} T(x,y,z,t) &= \frac{D}{k}\int_{\tau=0}^{t}\int_{x'=0}^{L}\int_{y'=0}^{W}\int_{z'=0}^{H}\sum_{l,m,n} e^{-D\left(\alpha_l^2+\beta_m^2+\gamma_n^2\right)\cdot(t-\tau)}\cdot \\ &\quad \frac{X_l(x)Y_m(y)Z_n(z)X_l(x')Y_m(y')Z_n(z')}{N_x(l)N_y(m)N_z(n)} g(x',y',z',\tau)dz'dy'dx'd\tau \\ &= \frac{D}{k}\int_{\tau=0}^{t}\int_{x'=0}^{L}\int_{y'=0}^{W}\sum_{l,m,n} e^{-D\left(\alpha_l^2+\beta_m^2+\gamma_n^2\right)\cdot(t-\tau)} \\ &\quad \frac{X_l(x)Y_m(y)Z_n(z)X_l(x')Y_m(y')Z_n(H)}{N_x(l)N_y(m)N_z(n)} q(x',y',\tau)dy'dx'd\tau. \end{aligned} \tag{24}$$

The integral operator and the summation operator can be interchanged. For a laser beam which moves from $(L/2, 0, H)$ to $(L/2, W, H)$ at a scanning speed v, the temperature can be rearranged as:

$$T(x,y,z,t) = \sum_{l,m,n}\left(\frac{D}{k}\frac{X_l(x)Y_m(y)Z_n(z)Z_n(H)}{N_x(l)N_y(m)N_z(n)}\int_0^t\int_0^W\int_0^L X_l(x')Y_m(y')q(x',y',\tau)e^{-D(\alpha_l^2+\beta_m^2+\gamma_n^2)\cdot(t-\tau)}dx'dy'd\tau\right). \tag{25}$$

With Eq. (25), temperature field can be integrated numerically via Quasi-Monte Carlo method with low-discrepancy sequences. Because of the existence of the exponential terms, as the eigenvalues increases, the summation converges rapidly.

Fig. 2 shows the comparison between the experimental data, the FEM simulation and the analytical solution. The results show that the proposed analytical model is consistent with both FEM model and experimental data.

4. Discussion

There are limitations about such analytical model. First, material properties were considered to be constant. Second, for the complexity induced by the Robin boundary condition, Eq. (2), a general solution could only be expressed as sums of integrations.

Last but not least, for an arbitrary domain, it commonly takes effort to calculate the corresponding eigenvalues. On the other hand, in the aspect of these limitations, FEM simulation is able to accommodate such complexities.

However, despite of these limitations, the proposed model can be used to calculate the temperature history at any given point quickly. The above FEM simulation with 48,000 elements costed about 5,234 s of total CPU (two 2.67 GHz Xeon processes with total 12 cores) time. On the contrary, evaluating the temperature at any given point and any given time can be done in the matter of seconds with a single core. Besides, the algorithm for Quasi-Monte Carlo method is mainly evaluating the integrand at each low-discrepancy sequences and then summing and averaging all the evaluations. Consequently, such algorithm can be programmed for a large scale parallelization.

5. Conclusion

A general solution for the transient temperature field in a finite slab, when heated by a moving laser heat source, was obtained by integral transform method. The eigenvalues were obtained by the method of separation of variables. Meanwhile an experiment and a FEM simulation were performed as verifications under a specific case that the power of the moving laser beam was considered to be uniformly distributed within the beam radius. The results showed that the analytical solution was consistent with both the experiment results and the FEM simulation.

Acknowledgments

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