Evolution of mixing width induced by general Rayleigh-Taylor instability

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Turbulent mixing induced by Rayleigh-Taylor (RT) instability occurs ubiquitously in many natural phenomena and engineering applications. As the simplest and primary descriptor of the mixing process, the evolution of mixing width of the mixing zone plays a notable role in the flows. The flows generally involve complex varying acceleration histories and widely varying density ratios, two dominant factors affecting the evolution of mixing width. However, no satisfactory theory for predicting the evolution has yet been established. Here a theory determining the evolution of mixing width in general RT flows is established to reproduce, first, all of the documented experiments conducted for diverse (i.e., constant, impulsive, oscillating, decreasing, increasing, and complex) acceleration histories and all density ratios. The theory is established in terms of the conservation principle, with special consideration given to the asymmetry of the volume-averaged density fields occurring in actual flows. The results reveal the sensitivity or insensitivity of the evolution of a mixing front of a neighboring light or heavy fluid to the degree of asymmetry and thus explain the distinct evolutions in two experiments with the same configurations.

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I. INTRODUCTION

As shown in Fig. 1, when two fluids of density $\rho_i$ (i = 1 = light, i = 2 = heavy) are separated by a perturbed interface and are accelerated in the direction opposite to that of the density gradient, Rayleigh-Taylor (RT) instability occurs and develops into turbulent mixing zones consisting of a bubble mixing zone (BMZ, formed when a light fluid penetrates a heavy fluid) and a spike mixing zone (SMZ, formed when a heavy fluid penetrates a light fluid) [1]. The mixing is a process by which two separated fluids of different densities seek to reduce their combined potential energy [2] and occurs ubiquitously [3] in systems extending from the micro- to astrophysical scales. As the simplest and primary descriptor of the mixing process, the location of the two edges of the mixing zone [i.e., mixing width $h_i(t)$, i = 1 = spikes, i = 2 = bubbles] plays a notable role [1] in many natural phenomena (e.g., supernova explosions [5]) and engineering applications (e.g., inertial confinement fusion [6]).

The scenarios generally involve complex varying acceleration histories $g(t)$ and widely varying density ratios $R \equiv \rho_2/\rho_1$, two dominant factors affecting the evolution of $h_i(t)$ in the turbulent regime [7].

To predict the evolution of $h_i(t)$ in the general RT problems, many models have been developed over the last few decades, such as the bubble-competition model [8], energy-transfer model [9], stationary-centroid model [10], and buoyancy-drag models [1,7,11–13]. However, no model has completely [7] reproduced the experimental observed $h_i(t)$ [12,14–16]. In fact, even for the simplest RT problem with a constant acceleration, i.e., $g = g_0$, previous models do not give satisfactory predictions. For example, the models produced only one pair of quadratic growth coefficients $\alpha_i^2 \equiv h_i/(Ag_i^2)$, while few pairs of $\alpha_i^2$ were observed [12] (see Fig. 2), where $A \equiv (R - 1)/(R + 1)$ ∈ (0, 1) is the Atwood number.

In this paper, we describe a theory for the general incompressible RT problem in an attempt to understand the problem. This theory is established in terms of the conservation principle, with special consideration given to the asymmetry of the volume-averaged density fields to highlight the actual flows occurring in nature. The theory is validated by the series of experiments [12,15,16]. Furthermore, it reveals that the evolution of $h_i(t)$ may be affected by initial perturbations, fluid properties, and degree of asymmetry, but governed essentially by mass, momentum conservation, and Newton’s second law.

II. RESULTS

A. Theory

In this section, we establish our theory by referring to the configuration and notations shown in Fig. 1. In Fig. 1, $\alpha = xyz$ is a noninertial reference frame fixed at the initially unperturbed interface [12], denoted by subscript 0. The acceleration is directed along the $y$ axis with $\ddot{y} = g(t)$, where the dot denotes the time derivative. Mixing is assumed to be statistically homogeneous in the $x$ and $z$ directions [7,10], with the cross-sectional area set to unity. Therefore, all of the ensemble-averaged quantities depend only on $y$ and $t$. The $h_i$ quantifies the distance of the interface to the mixing fronts, defined by the locations with concentration $c = 1\% (99\%)$ [17]. $V(y)$ quantifies the propagation speed of an iso-concentration surface [10] at $y$, with $V(h_i) \equiv v_i \equiv h_i$. Here $\ddot{f} \equiv \int_y^0 f(y')dy'$ quantifies the propagation speed of an iso-concentration surface [10] at $y$, with $V(h_i) \equiv v_i \equiv h_i$. $h_i$ essentially defines a volume average in SMZs and BMZs. From the left to the right, the flow fields consist of a pure light fluid zone with density $\rho_1$, a SMZ and BMZ with an average density $\bar{\rho}_1$ and $\bar{\rho}_2$, and a pure heavy fluid zone with density $\rho_2$. For the investigated RT problem, the density profile increases monotonically from $\rho_1$ of the light fluid, to $\rho_0$, at the interface, and to $\rho_2$ of the heavy fluid, leading to $\bar{\rho}_i = w_i\rho_0 + (1 - w_i)\rho_1$, where $w_i$ is positive and is named as mixing weight. Obviously $w_i$ should be less than the 1/2 of the mixing weight of the linearly varying
density profile since $\rho(y)$ transitions smoothly near $h_1$. With the aid of $w_i$, now we define the symmetry or asymmetry of volumed-averaged density fields for the currently investigated problem; density fields in BMZ and SMZ are said to be in symmetry (asymmetry) if the symmetry factor $\eta \equiv w_1/w_2 = (\neq) 1$. Based on this definition, we established our theory with conservation principles, as follows.

First, the conservation of mass [7] requires

$$\rho_1 h_1 + \rho_2 h_2 = \tilde{\rho}_1 h_1 + \tilde{\rho}_2 h_2.$$  

(1)

Second, given the success of the momentum-driven viewpoint [18] in understanding RT mixing and that of the stationary centroid hypothesis in predicting $h_i(t)$ of constant acceleration RT problems [7,10,19], it seems plausible for an approximation of the vanishing resultant force on the entire mixing region, resulting in the quasiconservation of momentum (similar to the stationary centroid hypothesis [7,10,19]):

$$\tilde{\rho}_1 h_1 V_1 = \tilde{\rho}_2 h_2 V_2,$$  

(2)

which is established by regarding the BMZ(SMZ) as a particle. Equation (2) implies that the evolutions of $h_1(t)$ and $h_2(t)$ depend on each other and thus should not be predicted with the two independent equations of the previous models [1,7,11–13]. Consequently one additional evolution equation is needed only. Given that the bubble structure is independent of the density ratio $R$ [8], we prefer to establish an evolution equation for $h_2(t)$ to avoid $R$-dependent parameters. For BMZ, it incurs three forces, namely, a buoyancy force $f_b = \rho_2 h_2 g(t)$, an inertial force $f_i = \tilde{\rho}_2 h_2 g(t)$, and a drag force $f_d = C_d \rho_2 v_2 |v_2|$. Therefore, the simplest physical assumption is the parabolic velocity profile of $V(y_1)/V(h_1) = (\eta y/h_1)^2$. The assumption gives $V_i = v_i/3$, with which the rearrangement [7] of Eqs. (1) to (3) yields the final evolution equation:

$$\gamma \chi v_i = \dot{v}_2 = \beta Ag(t) - C\phi v_2 |v_2|/h_2,$$  

(4)

where $\gamma = \tilde{\rho}_1/\tilde{\rho}_2 = [(R - 1)w_2\eta + (1 + \chi)\eta]/\Theta, \chi \equiv h_1/h_2$, $\beta \equiv 3(\phi - 1)/A = 3w_2 \chi \eta (R + 1) / \Theta, C \equiv 3C_d, \phi \equiv \rho_2/\rho_1 = R(1 + \chi) / \Theta$, and $\Theta \equiv R(1 + \chi \eta) + w_2 \chi \eta (1 - R).

B. Validation

Three parameters, $\eta, w_2, C$, are incorporated into the current theory. Due to the above mentioned $R$-independent bubble structure, the parameters are postulated to be $R$-independent and are determined definitely in the Appendix. With the determined parameters, our theory is systematically validated in Figs. 2–4 for general RT mixing.

FIG. 2. Validation of current theory for problems with constant acceleration and at all density ratios. The lines show the variations of the predicted quadratic growth coefficients $\alpha_i^A$ (red, upper)/$\alpha_i^L$ (black, lower), and its ratio $\alpha_i^A/\alpha_i^L$ (blue, inset) [see Eq. (A1) for the analytical expression] with $A$, or equivalently $R$. The symbols show the experimental results [12], and the two points with $\downarrow$ denote overmeasured data [12]. In the predictions, different values of $\eta$ are used to reveal the dependence of the evolutions on the degree of asymmetry, to reproduce the many observed pairs of $\alpha_i^A$ at the same $R$, and to explain the distinct difference between the empirical formula of $\alpha_i^A/\alpha_i^L \sim R^{0.33}$ (short dashed line) observed in linear electric motor experiments [12] (open diamonds) and that of $\alpha_i^A/\alpha_i^L \sim R^{0.23}$ (dash-dot line) in rocket-rig experiments [14,20] (see the references for the experimental data). Figure 2 shows the validation for an RT problem with constant acceleration, where our predictions (see the Appendix) are in good agreement with the results of the experiments [12] regardless of the density ratio. The figure further indicates that (1) except for the two overmeasured points, almost every observed $\alpha_i^A$ is within the region bounded by curves with the maximum and minimum $\eta$; (2) except for very few experiments, the density fields in BMZ and SMZ are symmetrical approximately in most cases since the majority of the observed $\alpha_i^A$ values lie on curves for which $\eta \approx 1, 3$; $\alpha_i^L(1), \text{ or equivalently } h_1(t)$, is closely (slightly) dependent on $\eta$, consistent with the results of the experiments; and (4) $\alpha_i^A/\alpha_i^L$ is closely dependent on $\eta$, explaining the distinct observed difference [20] between the results of Young’s [14] and Dimonte’s [12] experiments for the first time.
show the predictions for Eq. (4) with a symmetry factor in case d. The symbols show the experimental results, and the lines values of histories, as shown in the inset.

Figures 3–4 show a validation for problems with diverse acceleration histories and at all density ratios. For variable acceleration problems, although $C_g(t)$ has not yet been formulated, we can still predict $h(t)$ with a reasonable approximation of $C = \text{const}$ (see the Appendix), either entirely or piecewise. By means of this approximation, our theory is in good agreement with the results of experiments, and much better than the theory given in Ref. [7] (illustrated by the example in the inset of Fig. 4). Furthermore, as shown in Fig. 3(d), the good agreement between experiment and the exact solution to Eq. (4) for the special initial condition of $h_{10}/h_{20} = v_{10}/v_{20} = \alpha_2^A/\alpha_2^s$ [see Sec. II]. For general cases, Eq. (4) successfully reproduces all the evolutions of the impulsive experiments conducted by Dimonte [12], but only one example ($R = 23.4$, $F_68$, $G15$ [12]) is shown in the inset, which compares the results of the prediction [solid lines: Eq. (4); dashed lines: predictions by Ref. [7]] with those of the experiment [12] (symbols) for the evolutions of $h(t)$ (red), $h(t)$ (blue), and $h(t)$ (black). The current prediction was conducted by integrating Eq. (4) with initial values of $v_{10} = 0$ and $h_{10} = 10^{-6}m$ [7], experimentally measured $g(t)$ [12], and parameters $v_{10}, v_{20}, (g(t), \text{ and } C_{g0}(t))$, where a constant $C_{g0}(t)$ is adopted to enable the neglect of the variation in $C(g(t))$ during the extremely short duration of the impulsive acceleration stage I. In a very few experiments (especially those with a large $R$), which were probably affected by distinct initial perturbations, the density field was asymmetric, and thus $\eta$ needed to be adjusted slightly around 1, as in the case of the example shown in the inset for which $\eta = 1.1$.

Given the extremely short duration of stage I, we investigated only stage II and have denoted the quantities at the end of stage I with subscript $\delta$. For stage II with initial values of $v_{10}$ and $h_{10}$, Eq. (4) approximately has a power-law solution [7]

$$h_i = h_i \left( \frac{1}{v_{10} \theta_i^A} + 1 \right)^{\theta_i^A}, \quad t_i = h_i \frac{\theta_i^A}{v_{10} \eta i},$$

where $\varepsilon(t)$ is generally a time-dependent small quantity (see Ref. [7] for more information).

The solution given in Eq. (5) enables us to understand some long-standing questions. First, for the special initial condition of $x_A = h_{10}/h_{20} = v_{10}/v_{20} = \chi_0^A = \alpha_2^A/\alpha_2^s$, one can verify that $\varepsilon(t)$ equals zero exactly, with the corresponding power index of $\theta_i^A = \theta_i^A = \theta_i^A[1 + \varepsilon(t)]$ (see Ref. [7] for more information). In fact, substituting this exact power-law solution into the first equality of Eq. (4) yields $y = \chi_0^A = 1$ [similar to $y = \chi_0^A = 1$ obtained for problems with constant acceleration in the Appendix], from which the required special initial condition of $x_A = \chi_0^A$ is derived. This exact solution can explain the observed $\theta_i^A \approx \theta_i^A$ for $A < 0.8$ (see Fig. 4) as follows: for a problem with a small
or moderate density ratio and a positive acceleration history, previous studies [12,15] suggested an empirical formula of 
$$h_i \approx \alpha_i^A \left( \int \sqrt{g(t)} \, dt \right)^2$$, which, when applied to stage I, gives the special initial condition, leading to $$\theta_i^A \approx \theta_i^A$$. Second, for general cases, if $$t \gg t_\delta$$ and we neglect the high-order modification by $$\varepsilon(t)$$, we can obtain $$\theta_i^A \approx \theta_i^A$$ explicitly by following the procedures given by Dimonte [7], and $$\theta_i^A \approx f(\chi, t, \delta, t_\delta, \eta)$$ implicitly by substituting the power-law first equality of Eq. (4). The former reveals that $$\theta_i^A$$, or equivalently $$h_i(t)$$, is nearly independent of the initial conditions, and determined dominantly by $$\eta$$ because $$\phi(R, \eta, \chi(t))$$ is only slightly dependent on $$\chi$$ (this can be verified with the aid of the expression of $$\phi$$). In contrast, the latter implies that $$\theta_i^A$$, or equivalently $$h_1(t)$$, is sensitive to $$\eta$$ and the initial conditions, as verified by numerical integration. These conclusions are consistent with the experimental [12] and theoretical [21] results.

III. DISCUSSION

We give some discussions on the newly defined parameters of $$w_2$$ and $$\eta$$ involved in current theory. First, as pointed out in the beginning of the Sec. II B, the two parameters are postulated to be $$R$$-independent since the universality of bubble structure for all $$R$$. Second, according to the expression given in (A2), the $$w_2$$ depends only on $$\eta$$ by giving $$\alpha_2^0$$ and $$\alpha_1^0$$ (see the next paragraph on the determination of $$\alpha_2$$ and $$\alpha_1$$). The mixing weight $$w_2$$ characterizes the relative mass ratio of heavy fluid to light fluid in BMZ, and it can be determined from the cross section-averaged density profile. For the problem with constant acceleration, the current theory of $$w_2(\eta = 1, \alpha_2 = 0.06, \alpha_2^0 = 0.05) = 0.25 < 1/2$$ implies that the assumption of the linearly varied averaged density profile [7] is not a good approximation, and further study of the density profile is valuable. As for the symmetry factor $$\eta$$, it is the unique parameter in the current theory. The $$\eta$$ is introduced actually as a free parameter to reproduce, explain, and reveal the different evolutions in experiments using the same density ratio and acceleration history, and a strong (weak) dependence of $$h_{12}(t)$$ on $$\eta$$ is found. The value of $$\eta$$ may depend on many factors and thus is sensitive in different cases. However, in the sense of statistics, we argue that $$\eta$$ should be tend to unity and independent on $$R$$, a hypothesis coming from the random perturbations and the self-organized mixing. The good agreement of experiments and predictions shown in this paper confirms this hypothesis. Indeed, for the problem with either constant or impulsive acceleration, the good agreement of $$\alpha_1^A$$ (Fig. 2) and $$\theta_i^A$$ (Fig. 3) with $$\eta \approx 1$$ between the experiments and predictions implies that the density fields in BMZ and SMZ are symmetrically approximate in the sense of statistics. Moreover, for each specific case, the $$\eta$$ does not equal unity exactly in most experiments. This fact implies the symmetry of the volume-averaged density fields in BMZ and SMZ would break easily in actual flows, i.e., the universality of asymmetry in nature. Although $$\eta$$ may depend on many factors, we infer that the asymmetry of initial perturbations promise to be the most important factors. Overall, the quasi-symmetry of volume-averaged density fields in SMZ and BMZ (i.e., $$\eta \approx 1$$) are seem to be universal in actual flows, and current work further demonstrates the success of introducing the degree of asymmetry to quantify and understand the RT problem. For other problems, the idea may work, too.

Finally, our theory was validated systematically in terms of reproducing the results of all the available experiments, but only those results obtained for systems with immiscible inmiscid fluids [8,12,14,22,23] and natural perturbations [12,14,23] are presented in this paper. As is well known, however, $$\alpha_2$$ is highly dependent on the fluid properties (such as viscosity, miscibility) [7,17] and initial perturbations [20,24,25]. Therefore, for other systems with notably different media and/or perturbations, slightly different values of parameters $$\alpha_2$$ and/or $$\alpha_1$$ may be used. Nevertheless, the good agreements substantially confirm that the evolutions of general RT mixing are governed essentially by the conservation principle.

IV. CONCLUSIONS

A theory is established to predict the evolution of mixing width in general Rayleigh-Taylor flows. Different from the previous buoyancy-drag models, this theory is established by additionally considering mass and momentum conversation, by using the volumed-averaged speed of the advance of the mixing zone, by introducing a symmetry factor to highlight the property of density fields occurring in actual flows, and by assuming a parabolic-varied velocity profile. The parameters in the current theory are determined definitely by considering the asymptotic solutions. The theory is validated by the series of experiments with different acceleration histories and at any density ratios. To the best of our knowledge, this is the first time that a theory has successfully reproduced the results of all the experiments with the same parameters. The study further reveals that, besides the known dependence on fluid properties and initial perturbation, the time evolution of $$h_i(t)$$, especially $$h_1(t)$$, also depends on the symmetry factor $$\eta$$, but governed essentially by mass, momentum conservation, and Newton’s second law.

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APPENDIX: DETERMINATION OF PARAMETERS

Three parameters of $$\eta, w_2, C$$ are determined in four steps as follows.

Step I: Obtain the exact quadratic solution of Eq. (4) for the RT problem with constant acceleration $$g = go$$. For such a problem with initial values of $$h_0 = 0$$ and $$w_0 = 0$$, we can verify that Eq. (4) can solve (see Ref. [7] for more information)

$$h_i = \alpha_i^A A \sqrt{gt^2}, \quad \alpha_i^A = \beta/(2 + 4Cg, \phi), \quad \alpha_i^A = \chi_i^A \alpha_i^A, \quad \alpha_i^A$$, (A1)

where $$\chi_i^A$$, equivalent to $$\alpha_i^A/\alpha_i^A$$ or $$h_i/h_2$$, are determined as follows. Substituting the quadratic solution to the first equality of Eq. (4) yields $$\gamma(\chi_i^A)^2 = 1$$ and then $$\sum_{n=0}^{\infty} a_n(\chi_i^A)^n = 0$$, for which the positive real root gives $$\chi_i^A = \ldots$$

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Step II: Establish the algebraic interrelation between the parameters and the asymptotic quadratic-growth-coefficients with the solution obtained in step I. When $A \to 0$, we can obtain $\frac{R}{\chi_{\eta}(\eta)} \to \gamma \rightarrow 1$ and $\beta \to 6\omega_{2}\eta/(1+\eta)$. When $A \to 1$, we can first obtain $\chi_{\eta} = \left[(1-w^{1/2})+\sqrt{(1-w^{1/2})^{2}+4w_{2}\omega_{2}/(2w_{2}\eta)}\right]$ by using $A \to \infty$ and $\gamma \left(\chi_{\eta}^{2}\right) = 1$ (see step I), and then $\phi \to (1+\chi_{\eta}^{2})/\Psi, \gamma \rightarrow w_{2}\eta/\Psi$ with $\Psi = 1 + (1-w^{1/2})\chi_{\eta}^{2}$.

Consequently, the algebraic interrelations between the parameters of $\eta, w_{2}$, and $C$ and the asymptotic quadratic growth coefficients of $a_{\eta}^{1}, a_{\eta}^{2}, \alpha_{1}, \alpha_{2}$ can be obtained by substituting the above asymptotic results into the formula for $a_{\eta}^{1}$ in Eq. (A1). Omitting the complex algebraic operations, we obtain finally

\begin{align*}
\alpha_{\eta}^{1}\left(\eta,C_{\eta},w_{2}\right) & = 6w_{2}\eta/\left[(1+\eta)(2+4C_{\eta})\right] \\
\alpha_{\eta}^{2}\left(\eta,C_{\eta},w_{2}\right) & = 3w_{2}^{2}\chi_{\eta}^{2}/\left[2\Psi + 4C_{\eta}(1+\chi_{\eta}^{2})\right] \\
\alpha_{1}\left(\eta,C_{\eta},w_{2}\right) & = \alpha_{\eta}^{1}/\chi_{\eta}^{1} \\
w_{2}\left(\eta,a_{\eta}^{1},a_{\eta}^{2}\right) & = \left(T^{2} + 6\eta a_{\eta}^{1}T\right)/\left[36\eta(a_{\eta}^{1})^{2} + 6\eta a_{\eta}^{1}T\right] \\
C_{\eta}\left(\eta,a_{\eta}^{1},a_{\eta}^{2}\right) & = 3w_{2}\eta/\left[2a_{\eta}^{1}(1+\eta)\right] - 1/2 \\
\eta\left(\eta,a_{\eta}^{1},a_{\eta}^{2}\right) & = \left[-b_{1} - \sqrt{b_{1}^{2} - 4b_{1}b_{2}}\right]/(2b_{1}) \\
\eta\left(\eta,a_{\eta}^{1},a_{\eta}^{2}\right) & = \left[-d_{2} - \sqrt{d_{2}^{2} - 4d_{2}d_{1}}\right]/(2d_{1}),
\end{align*}

where $\chi_{\eta}^{1} = \left[(1-w_{1})\eta + ((1-w_{1})^{2}n^{2} + 4w_{2}\eta)/2w_{2}\eta\right]/(2w_{2}\eta), T = e_{1} + e\eta_{1}, b_{1} = c_{1}^{2} + e_{1}e_{3}(1-w_{2}), b_{2} = 2e_{1}e_{3} + e_{1}e_{2}(1-w_{2}), d_{2} = 2e_{1}e_{3}(\mu + 1) + c_{1}e_{2} - e_{2}e_{3}, d_{1} = c_{1}^{2}\mu(\mu + 1) - e_{1}^{2} - e_{1}e_{2}, e_{1} = a_{\eta}^{1}(3 + 2a_{\eta}^{2}), e_{2} = 6a_{\eta}^{2}, e_{3} = e_{1} - e_{2}$ and $\mu = a_{\eta}^{1}/a_{\eta}^{2}$.

Step III: Determine the values of the parameters. First, the range of $\eta \in \left(\eta_{\min}, \eta_{\max}\right)$ is determined by using the constraints of $w_{2}(\eta,a_{\eta}^{1},a_{\eta}^{2}) < 1/2$ (see Fig. 1) and $a_{\eta}^{1}(\eta,a_{\eta}^{0},a_{\eta}^{2}) < 1/2$ (restricted by free fall [7]) and imposing the asymptotic requirements of $a_{\eta}^{1}/a_{\eta}^{0} = 0.06$ (see the descriptions of the experiments [12,14,23]) and $a_{\eta}^{1}/a_{\eta}^{0} = 0.05$ (see other theories [8,10,22]).

In fact, the fourth (third) expressions in Eq. (A2) show that $w_{2}(\eta,a_{\eta}^{1})$ is a decreasing (increasing) function of $\eta$ in the region near $\eta = 1$, thus giving $\eta_{\min} = \eta(w_{2} = 1/2,a_{\eta}^{0},a_{\eta}^{2}) = 0.56$ and $\eta_{\max} = \eta(a_{\eta}^{1} = 1/2,a_{\eta}^{0},a_{\eta}^{2}) = 1.37$. Second, by using the fourth and fifth expressions of Eq. (A2), one can calculate $w_{2}$ and $C_{\eta}$ by assigning specific $\eta$ to give specifically $w_{2}(\eta = 1, a_{\eta}^{0}, a_{\eta}^{2}) = 0.24$ and $C_{\eta}(\eta = 1, a_{\eta}^{0}, a_{\eta}^{2}) = 2.5$, a recommended value given by Dimonte [7].

Step IV: Utilize the obtained $C_{\eta}$ to determine the drag coefficient $C_{\eta}$ for the variable acceleration problem. For variable acceleration problem, a time-dependent value of $C_{\eta}(t)$ is expected to be obtained with the main logic summarized as follows. In physics, drag is proportional to the surface area of the bubble structure [1,8] (denoted as $S$) and is highly dependent on the direction of mixing. For a case in which $\omega_{2} \geq 0$, the drag is dominated by chunk mixing near the local mixing front, and the experiments further imply a negative correlation between $S$ and $dS/dt$ [12,15], leading to $C = C_{\eta}(t)$, $C \approx C_{\eta}(t), C < C_{\eta}(t), C > C_{\eta}(t)$ for problems driven by constant, oscillating, increasing, and decreasing acceleration histories, respectively. In contrast, for cases in which $\omega_{2} < 0$, the drag is dominated by atomic mixing [26] across the entire BMZ and SMZ ($S > S_{\omega_{2} > 0}$), leading to $C > C_{\eta}$.