The influence of crack-orientation distribution on the mechanical properties of pre-cracked brittle media

Xiaguang Zeng, Yjie Wei*

LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, PR China

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Cracks in kerogen-rich shales and other brittle rock-like materials have a tremendous impact on their elastic properties and strength. In this paper, we investigate the effective mechanical properties of shale plates with pre-existing cracks. We employ the extended finite element method (XFEM) to investigate a pre-cracked medium with an elastic, isotropic and brittle shale matrix. We show how the mechanical properties of the orthotropic shale plates are dependent on the crack density and the standard deviation of crack angles. Both the Young's modulus and the Poisson's ratio of the cracked media exhibit a linear dependence on the standard deviation of crack angles, in contrast to the nonlinear dependence of the strength on the angle deviation. Finally, we propose mechanical models to capture the relationship between the mechanical properties and the distribution characteristics of pre-existing cracks in shales. These phenomenological models could be applied to estimate the fracturing behavior of shales in engineering practice.

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1. Introduction

Shale gas and shale oil are changing the world’s energy market. One key technology that exploits these resources is hydraulic fracturing, more commonly known as fracking (Bažant et al., 2014). To achieve effective fracking, engineers must thoroughly understand the macroscopic mechanical properties of kerogen-rich shales, including their elastic properties and fracture strength. Current knowledge about these elastic properties has been obtained from the interpretation of seismic waves propagating through a shale formation or the laboratory mechanical tests of shale drill cores. Ideally, we desire to construct a sound physical connection between the microstructure characteristics of shales and the macroscopic properties of the media, especially the pre-existing cracks in the media. Typically, shales display significant anisotropy (Sarker and Batzle, 2010; Sondergeld and Rai, 2011; Sone and Zoback, 2013a) and are brittle and transversely isotropic materials, with a symmetry axis vertical to their sedimentary plane (Vasin et al., 2013). The elastic anisotropy is partly caused by the preferred orientation of mineral components (Lonardelli et al., 2007). To address this anisotropy, theoretical works have been developed to predict effective elastic properties (Hornby et al., 1994; Lonardelli et al., 2007; Sayers, 2005; Vasin et al., 2013). Because natural fractures are common in shales and generally have a dominant trend (Gale et al., 2007), they largely contribute to the anisotropy (Vernik, 1993). Following fracking in shale formations, complex crack networks may be present together with the natural cracks; this has been regarded as a critical factor for economic or prospective production from shale reservoirs (Gale et al., 2014; Walton and McLennan, 2013). From this aspect, the interaction of fracking with pre-existing cracks in shales has attracted attention from the fields of solid mechanics, geophysics and composite materials in the past decades.

Theoretical methods have been developed to determine the effective elastic moduli of isotropic solids containing randomly oriented cracks, such as the self-consistent method (Budiansky and O’Connell, 1976), the general self-consistent method (Huang et al., 1994), the differential method (Hashin, 1988; Zimmerman, 1985), and the Mori-Tanaka method (Benveniste, 1987). In parallel, numerical methods have been broadly applied to calculate the effective mechanical properties of solids with pre-existing cracks, such as the finite element method (Makarynska et al., 2008; Shen and Li, 2004) and the boundary element method (Huang et al., 1996; Renaud et al., 1996). The differential method provides the closest estimation at low crack density, whereas the generalized self-consistent method or the non-interaction solution is more accurate than the other methods at high crack density.

The aforementioned methods all use a single parameter – the crack density – to characterize the random crack network and...
discussion properties. employed son's pronounced strength. cracks. fracture between derived based and has (2001) is vich, las vrical for Some ignored, al. for numerical approximations elastic properties authors was discovered angle for particularly elastic approximation elastic crack lengths tensile effective angle (2005) as Holder, et al., shale Planar transverse isotropy as described. For instance, some authors worked through two special cases, planar transverse isotropy and cylindrical transverse isotropy with circle cracks, and proposed formulas for the elastic moduli. Some authors (Feng et al., 2003; Gurevich, 2003; Huang et al., 1996; Laws and Brockenbrough, 1987; Thomsen, 1995; Wang et al., 2000; Zhan et al., 1999) later investigated similar situations using different methods. Except for in the case of the aligned crack situation, the crack angle effect is neglected, and the cracked solids are then regarded as macroscopic isotropic materials in the mentioned works. To the best of the authors’ knowledge, the crack angle effect on the elastic properties was first analyzed by Sevostianov and Kachanov (2001). The authors showed that the scatter of crack orientation has a pronounced effect on the effective properties of plasma-sprayed ceramic coatings. They developed a quantitative characterization for the microstructures of the coatings using a probability density function (Sevostianov et al., 2004). Later, Giordano and Colombo (2007a, 2007b) dealt with a similar situation and derived a theory for the elastic characterization of cracked solids based on a homogenization technique. Kushch et al. (2009) derived a series solution for the effective elastic moduli of anisotropic cracked materials. These works reported the crack angle effect on the elastic moduli at different crack densities. However, their results failed to explicitly and directly clarify the relationship between the mechanical properties and the crack angle distribution parameters.

In addition, the aforementioned works only calculated the effective elastic moduli. For shale fracturing design, the effective tensile strength is also a very important mechanical parameter. The fracture strength is affected by many factors, such as pressure (Lin, 1983) and lamination (Mokhtari et al., 2014). A strong correlation between the shale composition and the intact rock strength has been reported, particularly between the organic matter and the strength (Chong et al., 1982; Sone and Zoback, 2013b). Esene et al. (2007) discovered a logarithmic empirical relation between the tensile strength and temperature. Shales’ fracture strength is also reduced drastically by their cracks, which was demonstrated experimentally (Gale and Holder, 2008). Zhang et al. (1998) theoretically investigated the effects of the crack-length distribution and ligament sizes in the case of strongly interacting collinear cracks. Ma et al. (2005) used their numerical method to examine the influences of the crack distribution on the tensile strength. They found that the tensile strength exhibited a pronounced dependence on the distribution of crack orientations and crack locations as well as on the crack density. However, it appears that no study has thus far discovered the relationship between the general characterization of crack angles and the fracture strength for shales. Therefore, there is a compelling need to investigate the crack angle influence on the shales’ fracture strength.

In this paper, we examine the effects of crack angles on the mechanical properties of elastic brittle cracked shale plates. We employ numerical simulations to clarify the relationship between the crack angle distribution and the effective Young’s modulus, Poisson’s ratio, shear modulus and tensile strength. Furthermore, we propose approximation formulas that capture trends revealed in our numerical simulations. Section 2 describes the assumptions employed in this study and the model for effective mechanical properties. Section 3 presents the numerical results from extended finite element method (XFEM) simulations. Section 4 contains final discussions and concluding remarks.

2. Problem description

2.1. Crack distribution

As a representative shale matrix body, we consider an initially isotropic brittle linear elastic plate of area A with Young’s modulus E, Poisson’s ratio ν and tensile strength σs. This plate is permeated by N arbitrarily oriented straight cracks that do not intersect and whose centers are distributed homogeneously and randomly without overlapping. Each crack can be characterized by two variables: crack length li and crack angle θi (the angle between the crack plane and axis x1), as shown in Fig. 1. In the present work, we assume that these two random variables obey a truncated Gaussian distribution. The crack length li and the crack angle θi lie within the intervals li ∈ (0, +∞) and θi ∈ (0, π), respectively. Their probability density functions can be calculated by the following function (Johnson et al., 1995):

\[ f(x; \mu, s, c, d) = \begin{cases} \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x-c)^2}{2\sigma^2}\right), & c < x < d \\ 0, & x \leq c, x \geq d \end{cases} \]

where φ is the standard Gaussian probability density function, Φ is the cumulative distribution function, μ is the expectation, s is the standard deviation, and c and d bound the region of interest. We can calculate the expectation and standard deviation of the crack length or the crack angle by the following formulas (Johnson et al., 1995):

\[ \mu_4 = \mu + \frac{\phi(\mu)}{\Phi(\mu)} - \phi(\mu) \int \frac{1}{\Phi(\mu)} \, dx \]

\[ s_4 = \sqrt{\int \frac{(x-\mu)^2}{\Phi(\mu)} \, dx - \left( \int \frac{x}{\Phi(\mu)} \, dx - \mu \right)^2} \]

and herein, the truncated Gaussian distribution random variables describing the cracks are numerically simulated using the method proposed by Chopin (2010).

The assumed distributing cracks can be described by four distribution characteristic parameters, i.e., crack length expectation μ4, crack length standard deviation σ4, crack angle expectation μθ and crack angle standard deviation σθ. To simplify the analysis, we should first reduce the variables of the problem. According to the basic definitions of crack density ξ, crack length expectation μ4 and crack length standard deviation σ4, we may connect ξ with μ4 and σ4 with the following derivation. The ratio of cracked surface ξ is defined as ξ = \( \frac{1}{\sqrt{\pi}} \) \( \sum l \) \( \frac{1}{2} \) \( l \) \( N \), where A is the area of the cracked plate, N the total number of cracks, and li the length of the i-th crack for i=1, ..., N. It is straightforward to write the crack length expectation and its standard deviation as

\[ \mu_4 = \frac{1}{N} \sum_{i=1}^{N} l_i \] and \[ s_4 = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (l_i - \mu_4)^2} \]

and hence

\[ s_4^2 = \frac{1}{N} \sum_{i=1}^{N} l_i^2 - \frac{2\mu_4}{N} \sum_{i=1}^{N} l_i + \frac{1}{N} \sum_{i=1}^{N} \mu_4^2 \]
Since
\[
\frac{1}{N} \sum_{i=1}^{N} l_i^2 = \frac{4A}{N} \sum_{i=1}^{N} l_i^2 = \frac{4A}{N} \sum_{i=1}^{N} \left( \frac{l_i}{2} \right)^2 = \frac{4A}{N} \xi,
\]
and
\[
\frac{2\mu_{1L}}{N} \sum_{i=1}^{N} l_i = 2\mu_{1L} = 2\mu_{1L}^2
\]
we then have
\[
s_{1L}^2 = \frac{4A}{N} \xi - 2\mu_{1L}^2 + \mu_{1L}^2.
\]
Reformatting the above equation, we yield
\[
\xi = \frac{N}{4A} (\mu_{1L}^2 + s_{1L}^2).
\]
Note that Eq. (3) also holds for other crack distributions. The number of parameters for describing the crack distribution is decreased to three. To further simplify the problem, we assume that the plate is orthotropic after being permeated by the distributing cracks. This assumption is based on experimental observations from different groups that shale is approximately a vertical transversely isotropic solid (Hornby et al., 1994; Sayers, 1994; Sone and Zoback, 2013a; Vasin et al., 2013). We define the direction parallel to the crack angle expectation as principle direction 1, the corresponding perpendicular direction as principle direction 2 and the direction inclined at a 45° angle to direction 1 as direction x, as shown in Fig. 1. Then, the mechanical properties in the three directions will no longer depend on the parameter \(\mu_{2A}\). In summary, the mechanical properties of the orthotropic cracked plate in the three directions are dependent on only two crack parameters, viz., the crack density \(\xi\) and the crack angle standard deviation \(s_{1L}\).

2.2. Model for effective mechanical property calculations

To calculate the effective mechanical properties of the brittle orthotropic cracked plates, we simulate uniaxial stretching of the rectangular specimens. Fig. 1 illustrates a typical specimen, S, which comprises a central cracked area from the large cracked plate that was initially an isotropic linear elastic plate and later permeated by the above truncated Gaussian distributing cracks. There are three types of specimens that have different axis orientations with the same crack angle expectation. The first specimen type has its axes parallel to direction 1. The second type has its axes parallel to direction 2. The third type has its axes parallel to direction x. Each specimen is used to calculate the effective mechanical properties of the cracked plate in the ascribed direction. The length over width ratio of each Specimen S is 5, and its central cracked area is the product of length \(a\) and width \(b\).

If we exert stress \(\sigma\) on Specimen S, the average stresses \(\bar{\sigma}_{ij}\) and strains \(\bar{E}_{ij}\) of the specimen are connected by the effective compliance tensor \(\bar{S}_{ij}\), given by engineering constants as follows (Reddy, 2013):
\[
\begin{bmatrix}
\bar{E}_{11} \\
\bar{E}_{22} \\
\bar{E}_{12}
\end{bmatrix}
= 
\begin{bmatrix}
1/E_{11} & -v_{12}/E_{11} & 0 \\
-v_{12}/E_{11} & 1/E_{22} & 0 \\
0 & 0 & 1/G_{12}
\end{bmatrix}
\begin{bmatrix}
\bar{\sigma}_{11} \\
\bar{\sigma}_{22} \\
\bar{\sigma}_{12}
\end{bmatrix}.
\]
(4)

Considering the known symmetry condition \(E_1 v_{21} = E_2 v_{12}\) (Reddy, 2013), there are four independent variables in the effective compliance tensor, i.e., \(E_1\), \(E_2\), \(v_{12}\) and \(G_{12}\). We can directly determine the effective Young’s modulus and Poisson’s ratio in direction 1 and direction 2 by performing tension simulations of Specimen S in direction 1 and direction 2, respectively. We can obtain the corresponding Young’s modulus \(E_1\) and Poisson’s ratio \(v_{1x}\) using the information from the tensile response of the specimen in direction x. After determining the Young’s modulus and Poisson’s ratio in the three directions, we can approximately calculate the corresponding shear modulus \(G_{12}\) by the following formula derived from the off-axis tension tests (Morozov and Vasiliev, 2003):
\[
G_{12} = \frac{\sin^2 \beta \cos^2 \beta}{\frac{1}{E_1} - \cos^2 \beta \sin^2 \theta + \left( \frac{v_{12}}{E_1} + \frac{v_{12}}{E_1} \right) \sin^2 \beta \cos^2 \beta}.
\]
(5)

In this paper, Poisson’s ratio is so defined that the first subscript in \(v_{ij}\) \((i \neq j)\) corresponds to the load direction and the second subscript refers to the direction of the resulted lateral strain, which is commonly used in textbooks (Bower, 2009; Jones, 1998; Reddy, 2013) and is different from the definition used by (Morozov and Vasiliev, 2003; Vasiliev and Morozov, 2013).

We noticed that the off-axis tension tests should be used as an auxiliary way to determine the shear modulus. To determine the effective shear modulus directly, we add the simulations of the corresponding pure shearing specimen, Specimen R, which is also from the cracked plate, as shown in Fig. 1.

There are several models for crack extension in brittle media, such as the maximum principal tensile stress criterion (Erdogan and Sih, 1963), the strain energy density criterion (Sih, 1974),
the maximum energy release rate criterion (Palaniswamy and Knauss, 1978), and the modified strain energy density criterion (Theocaris and Andrianopoulos, 1982; Wei, 2012). We use a simple cohesive-like model to describe crack extension. The effective tensile strengths in the three directions $\sigma_{\parallel}$, $\sigma_{\perp}$ and $\sigma_{\sigma}$ are determined using the stretching simulations. In these simulations, damage initiation begins when the maximum principal stress $\sigma_{\text{max}}$ satisfies the relationship $1.0 \leq \frac{\sigma_{\text{max}}}{\sigma_{\text{tol}}} \leq 1.0 + f_{\text{SA}}$ (Simulia, 2011), where $\sigma_{\text{max}}$ represents the maximum allowable principal stress, $\sigma_{\text{tol}}$ the tolerance, and the symbol $\langle \rangle$ the Macaulay bracket with the usual interpretation; i.e., $\sigma_{\text{max}} = 0$ if $\sigma_{\text{max}} < 0$, and $\sigma_{\text{max}}$ if $\sigma_{\text{max}} \geq 0$. The damage evolution occurs once the initiation criterion is met. The linear damage evolution based on effective separation (Simulia, 2011) is used to model the fracture progress of the cracks. A scalar damage variable, $D$, representing the averaged overall damage at the intersection between the crack surfaces and the edges of cracked elements, is defined as $D = \frac{\delta_{\text{eff}}^0 (\sigma_{\text{max}} - \sigma_{\text{tol}})}{\delta_{\text{eff}}^m (\sigma_{\text{max}} - \sigma_{\text{tol}})}$

where $\delta_{\text{eff}}^0$ refers to the effective separation at failure, $\delta_{\text{eff}}^m$ the effective separation at the initiation of damage, and $\delta_{\text{eff}}^m$ the maximum value of the effective separation attained during the loading history. The damage variable $D = 0$ initially and then monotonically evolves from 0 to 1 upon further loading after the initiation of damage. The normal stress components $t_n$ (stress vector component perpendicular to the stress acting plane) and shear stress components $t_s$ (stress vector component parallel to the stress acting plane) are affected by the damage according to $t_n = (1 - D)t_n$ and $t_s = (1 - D)t_s$, where $T_n$ and $T_s$ are the normal and shear stress strengths without damage.

### 2.3. Numerical method

We adopt the XFEM for our numerical calculations to avoid remeshing crack surfaces and to thus decrease the computational cost drastically. The XFEM is a numerical technique based on the generalized finite element method and the partition of unity method (Moës et al., 1999); it is suitable for analyzing multiple crack growing problems (Budyn et al., 2004). The basic premise of the XFEM is to introduce the proper discontinuities inside the traditional finite elements via the partition of unity to relax the need for the mesh to conform to them. The major part of the method and its technical details have been reported elsewhere (Moës et al., 1999; Mohammadi, 2012; Stolarska et al., 2001).

For convenience, here, we briefly introduce the strategies of the XFEM applied in ABAQU (Simulia, 2011). In the ABAQUS XFEM, the enrichment functions used for the approximation for a displacement vector function are $\mathbf{u} = \sum_{i=1}^{n} N_i(x) \mathbf{u}_i + H(x) \mathbf{a}_i + \sum_{i=1}^{m} F_i(x) \mathbf{b}_i^e$, where $N_i(x)$ are the usual nodal shape functions, $\mathbf{u}_i$ is the usual nodal displacement vector associated with the continuous part of the traditional finite element solution, $\mathbf{a}_i$ is the nodal enriched degree of freedom (DOF) vector, $H(x)$ is the associated discontinuous jump function across the crack surfaces (called Heaviside function), $\mathbf{b}_i^e$ is the nodal enriched DOF vector, and $F_i(x)$ is the associated elastic asymptotic crack-tip functions. The discontinuous jump function $H(x)$ is given by $H(x) = \begin{cases} 1 & (\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{n} \geq 0 \\ -1 & \text{otherwise} \end{cases}$, where $\mathbf{x}$ is a Gauss point, $\mathbf{x}^*$ is the point on the crack closest to $\mathbf{x}$, and $\mathbf{n}$ is the unit outward normal to the crack at $\mathbf{x}^*$. The asymptotic crack tip functions are given by $F_{\alpha}(x) = (\sqrt{\sin \theta} \sin \frac{\theta}{2}, \sqrt{\cos \theta} \sin \frac{\theta}{2}, \sqrt{\sin \theta} \cos \frac{\theta}{2})$, where $(\alpha, \theta)$ is a polar coordinate system with its origin at the crack tip. Using the enriched displacement approximation $\mathbf{u}$, the discontinuities, such as cracks, can be analyzed in the framework of partition of unity and then incorporated into the conventional finite element method while retaining its properties such as sparsity and symmetry (Sukumar and Prévost, 2003). Notably, the additional DOFs $\mathbf{a}_i$ and $\mathbf{b}_i^e$ are only added to the enriched nodes. These nodes are used to describe the cracks by the level set method, which is a powerful numerical technique for analyzing and computing interface motion. In this method, the crack geometry is defined by two almost-orthogonal signed distance functions. The first describes the crack surface, while the second is used to construct an orthogonal surface such that the intersection of the two surfaces gives the crack front. Using the level set method, the moving of cracks is modeled based on the principles of linear elastic fracture mechanics and the phantom nodes or the cohesive segments method and the phantom nodes without remeshing.

### 3. Results and discussion

It is well-known that the crack density has a large influence on the effective mechanical properties of cracked solids. The self-consistent method, the general self-consistent method and the differential method all obtain accurate estimations of elastic moduli at low crack densities, such as at 0.05. At moderate crack densities, such as 0.25, there is only a small difference between their predictions. However, at large crack densities, such as 0.5, these three methods produce very different results. Considering these facts, three typical crack densities are considered in our present work, as shown in Table 1.

With the truncated Gaussian distribution in the range of $(0, \pi)$, the standard deviation of the crack angles $s_{\text{SA}}$ should reach a limit as the standard deviation $s_\theta$ of the corresponding standard Gaussian distribution approaches infinity. This limit is calculated numerically, not analytically, here by Eq. (2b), as shown in Table 2. The table shows that the limit angle is approximately $52^\circ$ for the case in which the cracked plate is isotropic at the macroscopic level because the crack angles are randomly distributed in all directions.

To provide statistically homogenous results, each specimen type is simulated with 8 different cracked areas having approximately 400–1000 cracks for a set of crack describing parameters ($\tilde{E}$, $s_\theta$); that is, 8 randomly cracked specimens are simulated in a calculation case. Table 3 provides details about the specimens used in the calculation cases. For the fixed crack densities shown in Table 1, we simulate 13 cases to study the crack angle effect on the effective mechanical properties. Detailed information for the individual cases is listed in Table 2. In each case, eight randomly cracked samples are generated for numerical simulations (Table 3). Thus, for one direction, each crack density requires the calculation of 208 specimens of type S and type R. There are a total of 624 specimens for the three directions in the presented work.

#### 3.1. Young’s modulus

After calculating the average stresses and strains at the cracked part of direction 1 stretching specimens (Specimen S), the values of the effective Young’s modulus in direction 1 are obtained according to the definition $E_1 = \frac{\bar{E}_1}{\overline{E}_1}$. Similarly, we obtain the Young’s modulus in direction 2 and direction $x$. Fig. 2 shows the effective Young’s modulus for all stretching specimens, and the error bar represents the standard deviations from eight independent simulations. As the angle standard deviation increases, the Young’s modulus in direction 1 decreases, the Young’s modulus in direction 2 increases, and the Young’s modulus in direction $x$ is almost constant.

<table>
<thead>
<tr>
<th>Crack density</th>
<th>0.05</th>
<th>0.25</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expectation of crack length (m)</td>
<td>0.0280</td>
<td>0.0436</td>
<td>0.0620</td>
</tr>
<tr>
<td>Standard deviation of crack length (m)</td>
<td>0.0056</td>
<td>0.0102</td>
<td>0.0143</td>
</tr>
</tbody>
</table>

| Table 1 Crack densities for the investigation of the crack angle effect on the mechanical properties of the cracked share plates. |
Table 2
Crack angle standard deviations of the truncated Gaussian distribution and the standard Gaussian distribution for the cracked shale plates.

<table>
<thead>
<tr>
<th>$s_{\Delta} (%)$</th>
<th>0</th>
<th>10</th>
<th>19.99</th>
<th>29.60</th>
<th>36.96</th>
<th>41.65</th>
<th>44.56</th>
<th>46.43</th>
<th>47.68</th>
<th>48.56</th>
<th>50.03</th>
<th>51.10</th>
<th>51.97</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{\Delta} (%)$</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>120</td>
<td>180</td>
<td>$1.8 \times 10^6$</td>
</tr>
</tbody>
</table>

$s_{\Delta}$: Crack angle standard deviation of the truncated Gaussian distribution, $s_{\Delta}$: Crack angle standard deviation of the standard Gaussian distribution.

Table 3
Sample details of the simulation specimens for a set of crack describing parameters ($\xi$, $s_{\Delta}$).

<table>
<thead>
<tr>
<th>No. of samples</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crack number (approximation)</td>
<td>1000</td>
<td>950</td>
<td>900</td>
<td>800</td>
<td>700</td>
<td>600</td>
<td>500</td>
<td>400</td>
</tr>
<tr>
<td>Length $a$ of cracked area (m)</td>
<td>2.0</td>
<td>1.9</td>
<td>1.8</td>
<td>1.6</td>
<td>1.4</td>
<td>1.2</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>Width $b$ of cracked area (m)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. 2. Influence of the crack angle standard deviation on the Young’s modulus of stretching specimens, (a) for crack density $\xi = 0.05$, (b) for crack density $\xi = 0.25$ and (c) for crack density $\xi = 0.5$. (d) Contours of stretching specimens showing the effect of the crack angle on the strains $(\varepsilon_y)$, where as the crack angle expectation increases, the average tensile strain increases with the same tensile stress, which explains the decrease of the effective Young’s modulus of the specimens. Likewise, the crack angle standard deviation causes the variation of the local tensile stiffness of the specimens and then the variation of their effective Young’s modulus.

As the angle standard deviation approaches the limit of 52°, the effective Young’s moduli in the three directions approach the same value at each crack density, which is essentially equal to the effective Young’s modulus of the isotropic cracked plates. Representative contours of the stretching specimens in directions 1, 2 and x in Fig. 2 (d) show that as the crack angle expectation increases, the area of bigger local tensile strains increases, which means a smaller local tensile stiffness and thus decreases in the effective Young’s modulus of the cracked specimens.

The results shown in Fig. 2 suggest that there is a nearly linear relationship between the effective Young’s modulus and the crack angle standard deviation; moreover, the Young’s modulus in direction $x$ is a constant. If the angle standard deviation is equal to zero, then the Young’s modulus in direction 1 is equal to the Young’s modulus of the uncracked matrix and the modulus in direction 2 is equal to the modulus of the plates with aligned cracks. Therefore, we can fit the relationship in Fig. 2 by a linear approximation, and the following approximation formulas are obtained for the ef-
The effective Young’s modulus of the isotropic cracked plates \( E_{\text{iso}} \) can be calculated by the differential approximation method. Here, the following formula derived from the generalized non-interacting bulk and shear modulus solutions (Shen and Li, 2004) is suggested:

\[
E_{\text{iso}} = E - \frac{S_{\text{A}}}{52} (E - E_{\text{iso}}), \tag{6a}
\]

\[
E_{x} = E_{\text{iso}}. \tag{6c}
\]

Meanwhile, Hashin’s differential formula (Shen and Li, 2004) is recommended to calculate the effective Young’s modulus of the aligned cracked plates:

\[
E_{\text{ali}} = \frac{E}{(1 + \pi \xi / 2)^4}. \tag{8}
\]

By substituting and simplifying, we obtain the following formulas, which can be used to determine the Young’s modulus of the orthotropic plates with truncated Gaussian uniform distributing cracks:

\[
E_{1} = E - E_{\text{A}} \frac{4 \pi \xi (4 v^2 + v - 3) - \pi^2 \xi^2 (6 v^2 + v + 7)}{78 (\pi \xi v - 3 \pi \xi - 4 v - 4)(\pi \xi v + \pi \xi - 2 v + 2)}, \tag{9a}
\]

\[
E_{2} = \frac{E}{(1 + \pi \xi / 2)^4} + E_{\text{A}} \frac{\pi^2 \xi^2 (15 v^2 - 4 v + 5) - 2 \pi \xi (25 v^2 + 4 v + 3) + 24 v^2 - 24}{3 (\pi \xi v - 3 \pi \xi - 4 v - 4)(\pi \xi v + \pi \xi - 2 v + 2)}, \tag{9b}
\]

\[
E_{x} = \frac{E}{(1 + \pi \xi / 2)^4} + \frac{E_{\text{A}}}{52} \frac{\pi^2 \xi^2 (15 v^2 - 4 v + 5) - 2 \pi \xi (25 v^2 + 4 v + 3) + 24 v^2 - 24}{3 (\pi \xi v - 3 \pi \xi - 4 v - 4)(\pi \xi v + \pi \xi - 2 v + 2)}. \tag{9c}
\]

The theoretical predictions are also shown in Fig. 2, which match reasonably well with the numerical results.

3.2. Poisson’s ratio

Using the numerical simulation results, the effective Poisson’s ratio of the cracked plates is calculated according to the standard
definition $v_{ij} = -	ilde{e}_{ij}/	ilde{e}_b$ for $i \neq j$, $\tilde{e}_b$ represents the strains in the direction of applied stresses and $\tilde{e}_{ij}$ the strains in the associated lateral direction. Hence the first subscript in $v_{ij}$ corresponds to the load direction and the second subscript refers to the direction of the resulted lateral strain. The Poisson’s ratio results of the stretching simulations are shown in Fig. 3.

We can see from Fig. 3 that at each crack density, the effective Poisson’s ratios in direction 1, direction 2 and direction x converge as the crack angle standard deviation approaches the limit of 52°. When the crack density is moderate or high, the Poisson’s ratio in direction x is not between the values in the other two directions and is larger than that in direction 2. More interesting, when the crack density is high, a negative Poisson’s ratio is found. The negative Poisson’s ratio results from crack opening under tension. In consideration of these findings, the crack angle has a complex effect on the Poisson’s ratio.

As proposed by Wash (Case, 1984), there is a simple linear relationship between the Young’s modulus and the Poisson’s ratio in brittle isotropic cracked materials. Using this relationship and the values of Poisson’s ratio at $s_{A}=52°$ shown in Fig. 3, the following Poisson’s ratio formula for the isotropic situation was obtained by fitting:

$$
v_{\text{iso}} = 2.2 \frac{E_{\text{iso}}}{E} - 1.1. \tag{10}
$$

There is also a similar linear relation between the Poisson’s ratio and the crack angle standard deviation in the orthotropic cracked plates when the crack density is not high. Therefore, we suggest the following formula for its estimation:

$$
v_{12} = v - \frac{s_{A}}{52} (v - v_{\text{iso}}). \tag{11}
$$

By substituting Eqs. (7) and (10) into Eq. (11), we finally obtain

$$

\begin{align}
&v_{12} = v - \frac{s_{A}}{52} (v - v_{\text{iso}}), \quad \text{(12a)} \\
&\text{On the other hand, there is a relationship due to the symmetry condition:} \\
&v_{21} = E_2 v_{12}/E_1. \quad \text{(12b)} \\
&\text{So after determining } E_1, E_2 \text{ and } v_{12}, \text{ one can calculate } v_{21} \text{ using the above equation.}
\end{align}
$$

3.3. Shear modulus

As mentioned in Section 2, pure shearing Specimen R is simulated to directly determine the effective shear modulus. Using the simulation results, we can easily obtain the shear modulus using the definition $G_{12} = \tilde{e}_{12}/\tilde{e}_{12}$. The shear modulus results are plotted in Fig. 4, and some typical contours are also shown. Fig. 4 illustrates that as the crack density increases, the area of bigger shear strains increases. This indicates a smaller local shear stiffness, which explains why the effective shear modulus of the specimens decreases.

The shear modulus of the cracked plates can be calculated by Eq. (5) using the stretching simulation results. Furthermore, according to $\beta = \pi/4$ and the symmetry condition $E_1 v_{21} = E_2 v_{12}$, Eq. (5) is simplified to the following formula:

$$
G_{12} = \frac{1}{\frac{s_{A}}{E_1} - \frac{s_{A}}{E_2} + \frac{s_{A}}{E_1} + \frac{2s_{A}}{E_2}}. \tag{13}
$$

The values of the effective shear modulus calculated by substituting the Young’s modulus and Poisson’s ratio of the stretching specimens into Eq. (13) are shown in Fig. 4, where $G = E_2/(1 + \nu)$. We notice that one should choose a suitable long specimen for the off-axis tension tests (Vasiliev and Morozov, 2013). As seen in Fig. 4, the length-to-width ratio for the stretching specimens adopted here is reasonable for a convergent result for the shear modulus. All these results suggest that the shear modulus is influenced by the crack density only and not by the crack angle standard deviation. Thus, the effective shear modulus can be calculated by an isotropic cracked plate’s formulas, such as the following differential solution (Shen and Li, 2004):

$$
G_{12} = \frac{G}{1 + \frac{1}{\frac{s_{A}}{E_1} + \frac{2s_{A}}{E_2}}}. \tag{14}
$$

3.4. Tensile strength

Strained specimens will fracture after their cracks begin to propagate because the material is assumed to be ideally brittle. Therefore, there is a peak stress in the stress-strain curve of the specimens with embedded cracks. This stress is selected as the effective tensile strength of the specimens. Fig. 5 shows the effective tensile strength of all the strained specimens. At each crack
density, the tensile strengths in the three directions approach a same value when the crack angle standard deviation approaches the limit of 52°. In the three crack density situations, the tensile strength in direction 1 decreases drastically, the tensile strength in direction 2 increases slowly, and the tensile strength in direction x almost remains constant. Compared to the crack angle influence on the Young’s modulus, the crack angle has a more significant impact on the tensile strength. A typical stress contour that occurs after crack propagation is shown in Fig. 5, which illustrates that some cracks begin to grow when the fracture criterion mentioned in Section 2.2 is satisfied.

From Fig. 5, it can be seen that the tensile strength is more sensitive to crack density and crack angle standard deviation than the Young’s modulus is to the two crack distribution parameters.
There are nonlinear relationships between the tensile strength and the crack density or the crack angle standard deviation. These relationships can be described using the idea of damage theory if proper damage variables are defined. According to Hult (1987), the following equations are selected to express these relationships:

\[ \sigma_{1a} = \sigma_1 \sqrt{1 - D_1}, \]  
\[ \sigma_{2a} = \sigma_2 \sqrt{1 - D_2}, \]  
\[ \sigma_{100} = \sigma_{18} = \sigma_1 \sqrt{1 - D_{100}}, \]  

where the three damage variables are defined by fitting the values shown in Fig. 5:  
\[ D_1 = \frac{\sqrt{2 \pi} \sigma_1 \cos(\pi/420)}{\sqrt{2 \pi} \sin(\pi/420) + 0.0038}, \]  
\[ D_2 = \frac{\sqrt{2 \pi} \sigma_2 \cos(\pi/420)}{\sqrt{2 \pi} \sin(\pi/420) + 0.0038}, \]  
\[ D_{100} = \frac{\sigma_{100} - \sigma_1}{\sqrt{2 \pi} \sin(\pi/420) + 0.0038}. \]

Note that the above damage variables cannot be used to predict the damage evolution because the cracked plates are supposed to be ideally brittle, and they fail immediately if their average tensile stresses approach the tensile strength.

4. Conclusions

In this paper, the crack angle effects on the Young's modulus, Poisson's ratio, shear modulus and tensile strength of cracked shale plates are studied using numerical simulations of stretching and shearing cracked specimens. These macroscopic mechanical properties are dependent on two crack distribution parameters, i.e., the crack density and the crack angle standard deviation, if the shale plates are orthotropic cracked materials. Via our numerical simulations, the following conclusions are drawn:

1. There is a linear relationship between the crack angle standard deviation and the effective Young's modulus. The effective Poisson's ratio of samples with low crack density appears to be linearly dependent on the crack angle standard deviation. In contrast, the effective shear modulus is influenced by the crack density but not the crack angle standard deviation.

2. The effective tensile strength is very sensitive to both the crack density and the crack angle standard deviation. There is a non-linear complex relationship between the tensile strength and the crack density as well as the crack angle standard deviation.

3. The effective Young's modulus, Poisson's ratio, and tensile strength in different directions are typically different. However, they converge as the crack angle standard deviation increases to its limit of 52°.

Combined with previously reported observations and theoretical analyses, we proposed several approximation formulas for the dependence of these mechanical properties on the crack statistical parameters, as shown in Eqs. (9), (12), (14) and (15). These constitutive models could be applicable as a first-order approximation to understand brittle and cracked shales.

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