Discontinuity Diagnosis Essentially Non-Oscillatory Schemes

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1 Introduction

Essentially non-oscillatory (ENO) schemes were first introduced by Harten, Engquist, Osher, and Chakravarthy [1] in the form of cell averages. Later, Shu and Osher [2] [3] developed ENO schemes applying the adaptive stencil idea to the numerical fluxes and using TVD Runge-Kutta type time discretizations.

Flux-version ENO schemes of Shu and Osher [3] are uniformly high order accurate right up to the shock wave, simpler to program and very robust to use. The stencil is designed to adapt in the vicinity of discontinuities to yield a one-sided interpolation if that becomes necessary. This gives an essentially non-oscillatory shock transition while maintaining an uniformly high order accuracy. However, there is also a drawback. ENO schemes have the inherent mechanism to compare discontinuities to chose the "smoothest" stencil, but they lack the mechanism to detect the discontinuities. In other words, when they approach two discontinuities at both sides, ENO schemes do not know there are two discontinuities and will choose the weaker discontinuity. This is the main reason why oscillation sometimes occurs and amplifies for ENO schemes with accuracy higher than three-order.

In this paper, we introduce a discontinuity diagnosis mechanism into the flux-version ENO schemes of Shu and Osher [3] and propose the discontinuity diagnosis essentially non-oscillatory (DDENO) schemes. The essence of this discontinuity diagnosis mechanism is to find out the discontinuities according to the information coming form neighboring stencils. DDENO schemes can detect the discontinuities at both sides automatically and cease to choose the weaker discontinuity. Thus essentially non-oscillatory property is achieved by this diagnosis mechanism around discontinuities and higher order accuracy is obtained by the upstream central schemes at the smooth regions.

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2 Derivation of DDENO schemes

In this section, we use the flux-version ENO schemes as the basis to formulate DDENO schemes. We choose the one-dimensional scalar conservation laws as an example:

$$L(u) = u_t. \tag{1}$$

Let us discretize the space into uniform intervals of size Δx. Let j be an integer, and let $x_j = j\Delta x$ denote cell centers and $x_{j+1/2}$ denote cell boundaries. Then we take the conservative schemes

$$L(u)_j = -\frac{1}{\Delta x}(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}), \tag{2}$$

where $\hat{f}_{j+1/2}$ is a higher-order numerical flux at the $x_{j+1/2}$ cell boundary.

We can actually assume $f(u)_x \geq 0$ for all u in the range of our interest. For a general flux, we can split it into two parts either globally or locally. Here we will only describe how $\hat{f}^{+}_{j+1/2}$ is computed on the basis of DDENO schemes. For simplicity, we will drop the "+" sign in the superscript. The formulas for the negative part of the split flux (with respect to $x_{j+1/2}$) are similar and will not be shown.

As we well known, the three-order ENO schemes work very well in almost all physical computations. It chooses one "smoothest" stencil from three candidate stencils and only uses the chosen stencil to approximate the numerical flux $\hat{f}_{j+1/2}$ at the cell boundary of $x_{j+1/2}$. Thus we use ENO-3 to construct the DDENO schemes. In the following parts, we use the phrase of "base stencil" to represent the "smoothest stencil of ENO-3" for the sake of simplicity.

(1) Compute the divided difference table of $f(u)$ up to rth desirable order.

(2) Construct the base stencil for each node. Choose the smoothest 3-point stencil for each node as the base stencil by comaparing the divided differences like ENO-3 schemes. Let us denote the base stencil at node j by $S(j)$:

$$S(j) = (k^j_{min}, k^j_{min} + 1, k^j_{max}), \tag{3}$$

where k^j_{min} and k^j_{max} are the left and right nodes of $S(j)$. The base stencil is chosen for each node, not for each cell boundary.

(3) Start ENO-3 schemes to approximate the numerical flux $\hat{f}_{j+1/2}$ at the cell boundary of $x_{j+1/2}$. Let us denote the ENO-3 stencil at $x_{j+1/2}$ by $S(j + 1/2)$:

$$S(j + 1/2) = (k_{min}, k_{min} + 1, k_{max}). \tag{4}$$

The discontinuity diagnosis mechanism is introduced into ENO-3 schemes here. Firstly, DDENO schemes analyzes the information coming from the left neighboring base stencil of node k_{min}. It compares the present stencil $S(j + 1/2)$ with the base stencil $S(k_{min})$ of node k_{min}. Secondly, it analyzes the information coming from the base stencil $S(k_{max})$ of the right adjacent node k_{max}.

(i) If

$$S(k_{min}) = S(j+1/2) = (k_{min}, k_{min}+1, k_{max}), \tag{5}$$

it means that there is a discontinuity between the nodes $k_{min}-1$ and k_{min}. Otherwise, it means that the left region of the present stencil $S(j+1/2)$ is smooth and the present stencil of ENO-3 can extend at least one point to the left.

(ii) If

$$S(k_{max}) = S(j+1/2) = (k_{min}, k_{min}+1, k_{max}), \tag{6}$$

it means that there is a discontinuity between the nodes k_{max} and $k_{max}+1$. Otherwise, it means that the right region of $S(j+1/2)$ is smooth and it can add at least one point to the right.

(iii) If

$$S(k_{min}) = S(k_{max}) = S(j+1/2) = (k_{min}, k_{min}+1, k_{max}), \tag{7}$$

it means that the present stencil $S(j+1/2)$ is surrounded by two discontinuities at both sides and DDENO ceases to increase points and just uses ENO-3 to approximate the numerical flux at $x_{j+1/2}$.

(iv) Otherwise, if

$$k_{min}(S(k_{min})) < k_{min} \; and \; k_{max}(S(k_{max})) > k_{max}, \tag{8}$$

DDENO extend ENO-3 to ENO-5.

(4) Inductively, DDENO repeats step (3) to increase the ENO-3 to higher order accuracy in smooth regions.

We remark that in the logical operation (iv), we use the logical operator "*and*". It is a very strong restriction and we do not need to know which nodes are chosen by $S(j+1/2)$. If we use the less strict logical operator "*or*", we should care about the condition of the present stencil. In some severe positions, the numerical flux is one-sidedly interpolated or extrapolated and high-order ENO schemes will induce oscillation. Therefore, the logical operator "*or*" should be carefully used. We do not suggest use higher order one-side extrapolation to approximate the numerical fluxes near discontinuities.

3 Numerical results

We consider here the classical Riemann problems for one-dimensional Euler system of gas dynamics for a polytropic gas. The time discretization was performed by third-order TVD Runge-Kutta-type methods developed by Shu and Osher [2]. All three examples were run with a CFL number of 0.6 and $\gamma = 1.4$. To solve the ordinary differential equation

$$\frac{du}{dt} = L(u), \tag{9}$$

where $L(u)$ is a discretization of the spatial operator, each physical flux was firstly split into there fluxes by Steger-Warming [5] flux vector splitting method with three eigenvalues of $u - a$, u, and $u + a$, where a is the sound velocity. Then Steger-Warming fluxes were further performed by the global Lax-Friedrichs (LF) splitting.

EXAMPLE 1. This is the well known Sod's problem [6]. The initial data are $(\rho_L, u_L, P_L) = (1, 0, 1)$, $(\rho_R, u_R, P_R) = (0.125, 0, 0.1)$. We use ENO-3 as the base stencil. We find that if equation (8) is satisfied, we can use ninth-order ENO schemes directly and do not need to repeat step (3) and (4). Interpolation of ninth-order accuracy is not necessary in physical simulations. We just want to examine the discontinuity diagnosis mechanism. Therefore, the accuracy of DDENO in this example is either three-order near discontinuities or ninth-order at smooth regions. The numerical results are presented in Fig. 1. By comparison, we can see that the shock wave and contact surface of DDENO are steeper than that of ENO-3. Also notice that the corners of rarefaction waves are better resolved.

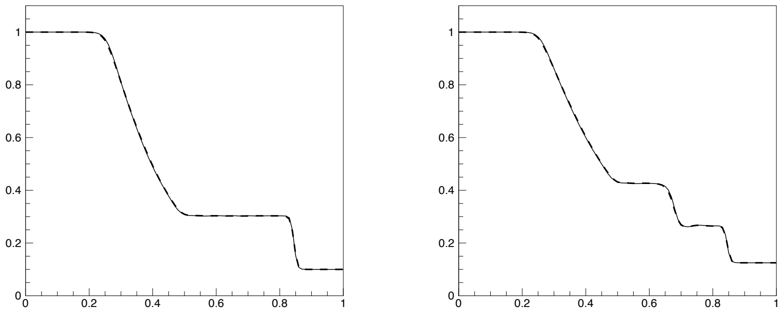

Fig. 1. DDENO-3-9 and ENO-3, Sod's problem, 100 points, t=0.2 (solid lines are DDENO-3-9 and dashed lines are ENO-3): (left) pressure; (right) density.

EXAMPLE 2. This is the Riemann problem proposed by Lax [7]. The initial data are $(\rho_L, u_L, P_L) = (0.445, 0.698, 3.528)$, $(\rho_R, u_R, P_R) = (0.5, 0, 0.571)$. As Sod's problem, we also use ENO-3 as the base stencil and ENO-9 at the smooth region. The results are shown in Fig. 2. Lax's problem is a tough test case for non-characteristic-based schemes of order at least three. Oscillations can easily appear for such schemes [4]. From Fig. 2 we can see that the density of DDENO is more steeper than that of eno-3.

EXAMPLE 3. Shu and Osher [3] presented a problem of a moving Mach 3 shock wave interacting with sine waves. This problem is a good model for the kinds of interactions that occur in simulations of compressible turbulence. The initial data are specified by

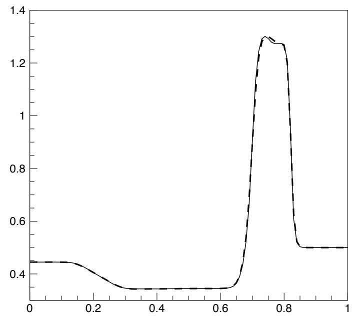

Fig. 2. DDENO-3-9 and ENO-3, Lax's problem, density, 100 points, t=0.13 (solid line is DDENO-3-9 and dashed line is ENO-3).

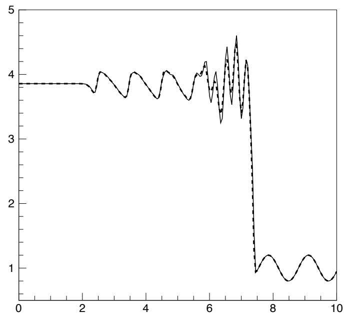

Fig. 3. DDENO-5-9 and ENO-5, Shu's problem, density, 200 points, t=1.8 (solid line is DDENO-5-9 and dashed line is ENO-5).

$$(\rho, u, P) = \begin{cases} (3.857, 2.629, 10.333), & x < 1.0 \\ (1 + \varepsilon \sin(5x), 0, 1), & x\varepsilon 1.0 \end{cases} \tag{10}$$

where $\varepsilon = 0.2$. The problem is run with 200 points in the interval [0, 10]. The results are plotted in Fig. 3. We mention above that the logical operator "or" in logical equation (8) should be used carefully. This problem looks severe but it is not difficult for ENO schemes to deal with because sine waves are periodically continuous functions. In this example, we use ENO-5 as the base stencil because the fluxes are one-side smooth. Then we use logical operator "and" to extend ENO-5 to ENO-9 at more smooth regions. Thus fifth-order one-sided interpolation or extrapolation is performed in this example. Therefore, the accuracy of the results in this example is fifth-order or ninth-order. From Fig. 3 we can see the expected improvements in resolution with higher orders.

4 Concluding remarks

In this paper, we introduce a discontinuity diagnosis mechanism into the flux-version ENO schemes and propose discontinuity diagnosis essentially non-oscillatory (DDENO) schemes. DDENO schemes use 3-point smoothest ENO stencil as the base stencil and detect discontinuities by comparing the present base stencil with neighboring base stencils. This mechanism prevents high-order ENO schemes from choosing the weaker discontinuities when it encounters two or more than two discontinuities at both sides. DDENO schemes are higher order accurate at smooth regions and three-order (the accuracy of the ENO base stencil) at discontinuities. Numerical experiments demonstrate that DDENO schemes work well in the examples and achieve higher order accuracy at smooth regions.

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