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General propagation lattice Boltzmann model for variable-coefficient non-isospectral KdV equation

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Abstract

In this paper, a general propagation lattice Boltzmann model for variable-coefficient non-isospectral Korteweg-de Vries (vc-nKdV) equation, which can describe the interfacial waves in a two layer liquid and Alfvén waves in a collisionless plasma, is proposed by selecting appropriate equilibrium distribution function and adding the compensate function. The Chapman-Enskog analysis shows that the vc-nKdV equation can be recovered correctly from the present model. Numerical simulation for the non-propagating one soliton of this equation in different situations is conducted as validation. It is found that the numerical results match well with the analytical solutions, which demonstrates that the current general propagation lattice Boltzmann model is a satisfactory and efficient method, and could be more stable and accurate than the standard lattice Bhatnagar-Gross-Krook model.

Keywords: General propagation lattice Boltzmann model; Variable-coefficient non-isospectral KdV equation; Numerical simulations; Soliton solutions

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1. Introduction

Nonlinear evolution equations (NLEEs) have been used to describe some nonlinear physical phenomena in several branches of science and engineering, e.g., hydrodynamics, plasma physics, elastic media and optical communication [1]. Investigation on various kinds of solutions of the NLEEs plays an important role in nonlinear science fields [1].

The lattice Boltzmann method (LBM) has been used in simulating some fluid flows [2], and extended to simulate some NLEEs, such as the nonlinear advection diffusion equation [3], the generalized nonlinear wave equations [4–7], and the coupled viscous Burgers' equation[8]. Unlike traditional numerical methods which discretize the governing equations in time and space, LBM is based on kinetic theory which tracks the dynamics of microcosmic particle ensembles [2]. Through the particle distribution function and equilibrium distribution function, the macroscopic variables are educed and the macroscopic equations are restored exactly [5].

Numerical studies for the NLEEs based on the LBM are generally about the constant-coefficient NLEEs. However, with the inhomogeneities of the media and non-uniformities of the boundaries considered, the variable-coefficient NLEEs can provide more realistic models than their constant-coefficient counterparts in modeling diverse phenomena [9]. Hence, in this paper, we focus on the variable-coefficient non-isospectral Korteweg-de Vries (vc-nKdV) equation, which can model the interfacial waves in a two-layer liquid and Alfvén waves in a collisionless plasma [10],

$$u_t + K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)\, u_x - h(t)(2u + x\, u_x) = 0\,, \tag{1}$$

where u is the wave-amplitude function of the scaled space coordinate x and time coordinate t, $K_0(t)$, $K_1(t)$ and $h(t)$ are all smooth functions of time t, and the subscripts x and t represent the spatial and temporal partial derivatives.

The remaining part of this paper will be structured as follows. General propagation lattice Boltzmann model (GPLB) for Eq. (1) will be derived in Section 2. Detailed numerical simulation for the non-propagating soliton of Eq. (1) will be performed in order to examine the accuracy and the stability of our model in Section 3. Finally, conclusions will be summarized in Section 4.

2. General propagation lattice Boltzmann model for Eq. (1)

For the one-dimension NLEE (1), the evolution law of the particle distribution function can be the corresponding discrete velocity Boltzmann equation [11] by introducing the Bhatnagar-Gross-Krook (BGK) collision operator [2], which is written as the following form,

$$\frac{\partial f_i}{\partial t} + \xi_i \frac{\partial f_i}{\partial x} = -\frac{1}{\tau_0}\left[f_i - f_i^{(eq)}\right] + F_i\,, \tag{2}$$

where $f_i(x,t)$ is a scalar function describing the particle distribution at position x and time t, $\{\xi_i,\, i = 0, 1, \ldots, n-1\}$ is the set of discrete velocities in the one-dimensional space with n different velocity directions ($D1Qn$) lattice model, $f_i^{(eq)}$ is the local equilibrium distribution function, τ_0 is the single relaxation time, and F_i is the source term. In this paper, we use the $D1Q5$ velocity model, where the discrete velocities can be defined as $\overrightarrow{\xi} = \{0,\, c,\, -c,\, 2c,\, -2c\}$, where $c = a\Delta x/\Delta t$ is a scale constant factor, Δx and Δt are represented the lattice space and time step, respectively, and a is a free parameter to adjust the propagation step of our model.

Eq. (2) can be decomposed into the collision and propagation steps [12] by applying the time-splitting method,

$$\frac{\partial f_i}{\partial t} = -\frac{1}{\tau_0}\left[f_i - f_i^{(eq)}\right] + F_i, \tag{3a}$$

$$\frac{\partial f_i}{\partial t} + \xi_i \frac{\partial f_i}{\partial x} = 0, \tag{3b}$$

and solved sequentially at each time step. The advantage of this replacement is that the collision and propagation steps can be treated with different numerical schemes, respectively.

For the collision equation (3a), there is no spatial derivative involved. Hence, the explicit Euler scheme is used to discretize Eq. (3a) into the following forms,

$$f_i^+(x,t) = \left(1 - \frac{1}{\tau}\right) f_i(x,t) + \frac{1}{\tau} f_i^{(eq)}(x,t) + \Delta t G_i(x,t) + \Delta t F_i, \tag{4}$$

where $\tau = \tau_0/\Delta t$ is the dimensionless relaxation time, and the correction term $G_i(x,t)$ is introduced into the collision step to eliminate the effect of the additional term [12]. It should be noted that the collision process is the same as that in the standard lattice Bhatnagar-Gross-Krook (SLBGK) models.

For the propagation equation (3b), we adopt an explicit two-level, three-point scheme [12] to discretize it,

$$f_i(x, t+\Delta t) = p_0 f_i^+(x,t) + p_{-1} f_i^+(x - L_i, t) + p_1 f_i^+(x + L_i, t), \quad L_i = \Delta x \cdot e_i, \tag{5}$$

where $p_0 + p_{-1} + p_1 = 1$ and $p_{-1} - p_1 = a = \Delta t \cdot \xi_i / L_i$. One solution of the above constraint can be expressed as follows,

$$p_0 = 1 - q, \quad p_{-1} = \frac{q+a}{2}, \quad p_1 = \frac{q-a}{2}, \tag{6}$$

where q is also one introduced free parameter. Clearly, the propagation process of our general model is different with that in the standard LBGK models, which is just a special case, i.e., $a = q = 1$. Based on the stability analysis in the numerical stability condition, these two parameters should satisfy $a^2 \leqslant q \leqslant 1$.

From the above, the combination of the collision scheme given by Eq. (4) and the propagation scheme given by Eq. (5) constructs the general propagation lattice Boltzmann model.

In the following, we will apply the multi-scale Chapman-Enskog [13] and Taylor expansions to obtain the specific expressions of the local equilibrium distribution function $f_i^{(eq)}$ and the correction terms G_i which will be used to complete our GPLB model for Eq. (1).

Firstly, applying the Taylor expansion to $f_i^+(x + L_i, t)$, $f_i^+(x - L_i, t)$ and $f_i^+(x,t)$, retaining the terms up to $O(\Delta t^4)$, and substituting them into Eq. (5), one can obtain that

$$\begin{aligned} f_i(x,t) + \Delta t \partial_t f_i(x,t) + \frac{\Delta t^2}{2}\partial_t^2 f_i(x,t) + \frac{\Delta t^3}{6}\partial_t^3 f_i(x,t) = f_i^+(x,t) - \Delta t\left(\xi_i \cdot \partial_x\right) f_i^+(x,t) \\ + \frac{\Delta t^2 q}{2a^2}\left(\xi_i \cdot \partial_x\right)^2 f_i^+(x,t) - \frac{\Delta t^3}{6a^2}\left(\xi_i \cdot \partial_x\right)^3 f_i^+(x,t) + O(\Delta t^4). \end{aligned} \tag{7}$$

Secondly, applying multi-scale Chapman-Enskog expansion up to the third-order in time t, the first-order in space x, the local particle distribution function f_i, the correction terms G_i and the source terms F_i can be expressed as,

$$\partial_t = \epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \epsilon^3\partial_{t_3}, \quad \partial_x = \epsilon\partial_{x_1}, \tag{8a}$$

$$f_i = \sum_{n=0}^{\infty} \epsilon^n f_i^{(n)} = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \epsilon^3 f_i^{(3)} + \ldots, \tag{8b}$$

$$F_i = \epsilon F_i^{(1)}, \quad G_i = \epsilon G_i^{(1)} + \epsilon^2 G_i^{(2)} + \epsilon^3 G_i^{(3)}. \tag{8c}$$

where ϵ is a small expansion parameter.

Substituting Eqs. (8a)-(8c) into Eq. (7) and coupling with Eq. (4), we can obtain series of differential equations for the first three orders of ϵ,

$$O(\epsilon^0): \;\; f_i^{(0)} = \left(1-\frac{1}{\tau}\right) f_i^{(0)} + \frac{1}{\tau} f_i^{(eq)}, \;\; \text{i.e.,} \;\; f_i^{(0)} = f_i^{(eq)}, \tag{9a}$$

$$O(\epsilon^1): \;\; f_i^{(1)} + \Delta t\,\partial_{t_1} f_i^{(0)} = \left(1-\frac{1}{\tau}\right) f_i^{(1)} + \Delta t\left(F_i^{(1)} + G_i^{(1)}\right) - \Delta t\,(\xi_i\cdot\partial_{x_1}) f_i^{(0)}, \tag{9b}$$

$$\begin{aligned} O(\epsilon^2): \;\; & f_i^{(2)} + \Delta t\left[\partial_{t_1} f_i^{(1)} + \partial_{t_2} f_i^{(0)}\right] + \frac{\Delta t^2}{2}\partial_{t_1}^2 f_i^{(0)} = \left(1-\frac{1}{\tau}\right) f_i^{(2)} + \Delta t G_i^{(2)} \\ & - \Delta t\,(\xi_i\cdot\partial_{x_1})\left[\left(1-\frac{1}{\tau}\right) f_i^{(1)} + \Delta t\left(F_i^{(1)} + G_i^{(1)}\right)\right] + \frac{\Delta t^2 q}{2a^2}(\xi_i\cdot\partial_{x_1})^2 f_i^{(0)}, \end{aligned} \tag{9c}$$

$$\begin{aligned} O(\epsilon^3): \;\; & f_i^{(3)} + \Delta t\left[\partial_{t_1} f_i^{(2)} + \partial_{t_2} f_i^{(1)} + \partial_{t_3} f_i^{(0)}\right] + \frac{\Delta t^2}{2}\left[\partial_{t_1}^2 f_i^{(1)} + 2\partial_{t_1,t_2}^2 f_i^{(0)}\right] + \frac{\Delta t^3}{6}\partial_{t_1}^3 f_i^{(0)} \\ & = \left(1-\frac{1}{\tau}\right) f_i^{(3)} + \Delta t G_i^{(3)} - \Delta t\,(\xi_i\cdot\partial_{x_1})\left[\left(1-\frac{1}{\tau}\right) f_i^{(2)} + \Delta t G_i^{(2)}\right] \\ & + \frac{\Delta t^2 q}{2a^2}(\xi_i\cdot\partial_{x_1})^2\left[\left(1-\frac{1}{\tau}\right) f_i^{(1)} + \Delta t\left(F_i^{(1)} + G_i^{(1)}\right)\right] - \frac{\Delta t^3}{6a^2}(\xi_i\cdot\partial_{x_1})^3 f_i^{(0)}, \end{aligned} \tag{9d}$$

Similar with the general LBM, and considering the conservation law of local mass, we define the macroscopic physical quantity u as distribution function

$$u(x,t) = \sum_i f_i(x,t) = \sum_i f_i^{(eq)}(x,t), \tag{10}$$

From Eq. (9a), one can obtain

$$\sum_i f_i^{(0)}(x,t) = u(x,t), \quad \sum_i f_i^{(n)}(x,t) = 0, \quad n > 0. \tag{11}$$

Substituting the lower order equations of ϵ into the higher order equations, we can simplify Eq. (9) as the following forms,

$$f_i^{(1)} = \tau\Delta t\left[\left(F_i^{(1)} + G_i^{(1)}\right) - \left(\partial_{t_1} + \xi_i\cdot\partial_{x_1}\right) f_i^{(0)}\right], \tag{12a}$$

$$-\frac{f_i^{(2)}}{\tau\Delta t^2} = \left[\partial_{t_2} + \left(\frac{1}{2}-\tau\right)\partial_{t_1}^2 + (1-2\tau)\left(\xi_i\cdot\partial_{x_1}\right)\partial_{t_1} - \left(\tau - 1 + \frac{q}{2a^2}\right)\left(\xi_i\cdot\partial_{x_1}\right)^2\right] f_i^{(0)} + \tau\left(\partial_{t_1} + \xi_i\cdot\partial_{x_1}\right)\left(F_i^{(1)} + G_i^{(1)}\right) - G_i^{(2)}, \tag{12b}$$

$$-\frac{f_i^{(3)}}{\tau\Delta t^3} = \left\{\left[\partial_{t_3} + \Delta t\,(1-2\tau)\,\partial_{t_1,t_2}^2 + \Delta t^2\left(\tau^2 - \tau + \frac{1}{6}\right)\right] + \Delta t\left[2\Delta t\left(\tau^2-\tau+\frac{1}{6}\right)\partial_{t_1}^2 + (1-2\tau)\,\partial_{t_2}\right]\left(\xi_i\cdot\partial_{x_1}\right) + \Delta t^2\left[\frac{q}{a^2}(\tau-\frac{1}{2}) + 3\tau^2 - 4\tau + 1\right]\partial_{t_1}\left(\xi_i\cdot\partial_{x_1}\right)^2 + \Delta t^2\left[\tau^2 + \left(\frac{q}{a^2}-2\right)\tau + 1 + \frac{1}{6a^2} - \frac{q}{a^2}\right]\left(\xi_i\cdot\partial_{x_1}\right)^3\right\} f_i^{(0)} + \tau\Delta t\left\{\left[\Delta t\left(\frac{1}{2}-\tau\right)\partial_{t_1}^2 + \partial_{t_2}\right] + \Delta t\,(1-2\tau)\,\partial_{t_1}\left(\xi_i\cdot\partial_{x_1}\right) + \Delta t\left(1 - \frac{q}{2a^2} - \tau\right)\left(\xi_i\cdot\partial_{x_1}\right)^2\right\}\left(F_i^{(1)} + G_i^{(1)}\right) + \tau\Delta t\left(\partial_{t_1} + \xi_i\cdot\partial_{x_1}\right) G_i^{(2)} - G_i^{(3)}. \tag{12c}$$

In order to recover Eq. (1), some constraints on equilibrium distribution function and correction terms are imposed as follows,

$$\sum_i \left(\xi_i,\, \xi_i^2,\, \xi_i^3\right)^T\cdot f_i^{(0)} = \left(0,\, 0,\, \frac{K_0(t)\,u}{\left[\tau^2 - \left(2 - \frac{q}{2a^2}\right)\tau + 1 + \frac{1}{6a^2} - \frac{q}{a^2}\right]\Delta t^2}\right)^T, \tag{13a}$$

$$\sum_i F_i = \sum_i \epsilon\, F_i^{(1)} = h(t)\,u, \tag{13b}$$

$$\sum_i G_i = 0\,, \quad \Rightarrow \quad \sum_i G_i^{(1)} = \sum_i G_i^{(2)} = \sum_i G_i^{(3)} = 0\,, \tag{13c}$$

$$\sum_i \xi_i\, G_i = \epsilon\sum_i \xi_i\, G_i^{(1)} + \epsilon^2\sum_i \xi_i\, G_i^{(2)} = \frac{1}{\tau\Delta t}\left[3K_0(t)\,u^2 + (4K_1(t) - h(t)x)\,u\right], \tag{13d}$$

$$\sum_i \xi_i^2\, G_i = \epsilon\sum_i \xi_i^2\, G_i^{(1)} = 0. \tag{13e}$$

It should note that every discrete lattice velocity ξ_i multiplied in the term of the derivatives is accompanied with ∂_x or ∂_{x_1}, which increases the order of this term to the small expansion parameter ϵ. While the order of the correction terms G_i to ϵ is fixed at third-order, leading the equalities in Eqs. (13c)-(13e).

Summing Eq. (12) over i and substituting Eqs. (13a)-(13e) into these equations, one can obtain the following forms,

$$\partial_{t_1} u = \frac{1}{\epsilon}\, h(t)\, u, \tag{14a}$$

$$\partial_{t_2} u + \frac{1}{\epsilon^2}\,\tau\Delta t\,\partial_x\left(\epsilon\sum_i \xi_i\, G_i^{(1)}\right) = 0, \tag{14b}$$

$$\partial_{t_3} u + \frac{1}{\epsilon^3}\tau\Delta t\,\partial_x\left(\epsilon^2\sum_i G_i^{(2)}\right) + \frac{1}{\epsilon^3}K_0(t)\,u_{xxx} = 0. \tag{14c}$$

Hereby, by taking Eq. (14a)×ϵ+Eq. (14b)×ϵ^2+Eq. (14c)×ϵ^3, we have,

$$\left(\epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \epsilon^3\partial_{t_3}\right)\,u + \partial_x\left[\tau\Delta t\sum_i \xi_i\,G_i\right] + K_0(t)\,u_{xxx} = h(t)\,u. \tag{15}$$

Coupling with the definition (8) and Eq. (13d), the above equation (15) can be transformed into the macroscopic equation (1),

$$u_t + 6K_0(t)uu_x + 4K_1(t)\,u_x + K_0(t)\,u_{xxx} - h(t)\,(u + xu_x) = h(t)\,u\,. \tag{16}$$

Considering the symmetry of the lattice velocity and regarding velocity-0 as the dominant term, we assume that the equilibrium distribution function satisfies the following constraints by a small free parameter σ, i.e., $\sigma f_0^{(0)} = \sum_{\alpha=1}^4 f_\alpha^{(0)} = f_1^{(0)} + f_2^{(0)} + f_3^{(0)} + f_4^{(0)}$. Besides, we can assume $G_1 = G_2 = -G_3$, and the source term is the same at each direction, i.e, $F_i = h(t)\,u/5$. Under these additional constraints and coupling with Eqs. (11)-(13e), one can obtain the specific expressions of the local equilibrium distribution functions $f_i^{(0)}$, correction terms G_i and source terms F_i,

$$f_i^{(0)} = \begin{cases} \dfrac{1}{1+\sigma}u, & i\text{=}0\ , \\ \left[\dfrac{2\sigma}{3(1+\sigma)} - \dfrac{\mathscr{F}}{6}\right]u, & i\text{=}1\ , \\ \left[\dfrac{2\sigma}{3(1+\sigma)} + \dfrac{\mathscr{F}}{6}\right]u, & i\text{=}2\ , \\ -\left[\dfrac{\sigma}{6(1+\sigma)} - \dfrac{\mathscr{F}}{12}\right]u, & i\text{=}3\ , \\ -\left[\dfrac{\sigma}{6(1+\sigma)} + \dfrac{\mathscr{F}}{12}\right]u, & i\text{=}4\ , \end{cases} \quad G_i = \begin{cases} \dfrac{1}{2}\mathscr{G}, & i\text{=}0\ , \\ -\dfrac{1}{3}\mathscr{G}, & i\text{=}1\ , \\ -\dfrac{1}{3}\mathscr{G}, & i\text{=}2\ , \\ \dfrac{1}{3}\mathscr{G}, & i\text{=}3\ , \\ -\dfrac{1}{6}\mathscr{G}, & i\text{=}4\ , \end{cases} \quad F_i = \frac{h(t)\,u}{5}, \tag{17}$$

where

$$\mathscr{F} = \frac{K_0(t)\Delta t}{\left[\tau^2 - \left(2 - \frac{q}{2a}\right)\tau + 1 + \frac{1}{6a^2} - \frac{q}{a^2}\right](a\Delta x)^3}, \quad \mathscr{G} = -\frac{\left[3K_0(t)\,u^2 + (4K_1(t) - h(t)x)\,u\right]}{a\tau\Delta x}\ .$$

3. Numerical simulation for the non-propagating soliton

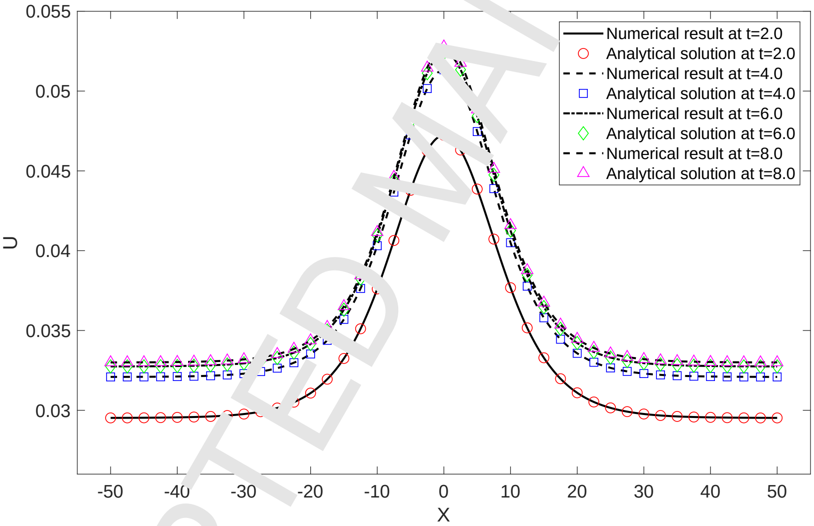
Validation of our work is conducted by comparing the analytical solutions and the numerical results. The global relative error **GRE** are used to measure the accuracy of our model [7], whose specific expressions are omitted here.

A non-propagating one-soliton solutions for the vc-nKdV equation (1) with time varying nonvanishing boundary condition in Ref. [10] are given by

$$\begin{aligned} &u(x,t) = L(t) + 2\eta^2\text{sech}^2(\vartheta), \quad \vartheta = \frac{f(t)}{2} - \eta x, L(t) = L(0)\exp\Big[\int_0^t 2h(t)dt\Big], \\ &\eta = \eta(0)\exp\Big[\int_0^t h(t)dt\Big], \quad f(t) = 8\int_0^t \eta\big[K_0(t)\eta^2 + \frac{3}{2}K_0(t)L(t) + K_1(t)\big]dt, \end{aligned} \tag{18}$$

In this simulation, we set $K_0(t) = 0.1, K_1(t) = 0$, $h(t) = 2t/[3(1+t^2)(2+t^2)]$, $L(0) = 1/30$ and $\eta(0) = 0.1$. Following the discussion procedures in Ref. [14], we can find that $\sigma > 0$ can be considered as an equilibrium parameter, which can adjust the amplitude in the process of the evolution, and set $\sigma = 0.05274$ in this simulation. The specific discussions are omitted here due to the limitations of this letter. Some other parameters are $a = 0.8$, $q = 0.64$, $\tau = 1.1686$,, $\Delta x = 0.05$, $c = 80$ (i.e., the corresponding time step is $\Delta t = 0.0005$) and the computation domain is fixed on $I = [-50, 50]$. Comparisons between detailed numerical results and analytical solutions are presented in Fig. 1. It can be found that the numerical results agree with the analytical solutions well.

Besides, the **GRE** of the numerical results obtained by SLBGK and GPLB models with different τ at $t = 2.0$ are presented in Tab. 1. We can find that the numerical results obtained by GPLB match with the analytical solutions well, while the SLBGK results get divergency at $\tau = 1.145$. This phenomenon implies that the present GPLB model is a satisfactory and efficient method, and could be more stable and accurate than SLBGK.



Figs. 1. Numerical result and analytical solution at $t = 2.0, 4.0, 6.0, 8.0$

GRE	$\tau = 1.145$	$\tau = 1.1686$	$\tau = 1.2$
$a = 0.8$, $q = 0.64$	1.3567e-003	2.6915e-005	3.1707e-005
SLBGK	$\star$	4.0898e-005	8.1698e-004

Table 1. **GRE** of the numerical results obtained by SLBGK and GPLB models with different τ at $t = 2.0$. ($\star$ represents the divergency.)

4. Conclusions

In this paper, a general propagation lattice Boltzmann model for variable-coefficient non-isospectral Korteweg-de Vries equation (1), has been proposed through selecting equilibrium distribution function and adding the compensate function, appropriately. The D1Q5 velocity model has been used in numerical simulations with different forms of Eq. (1). Through the Chapman-Enskog analysis, it has been found that Eq. (1) can be recovered correctly from our present general propagation lattice Boltzmann model. Numerical simulation for the non-propagating one soliton of this equation in different situations has been conducted as validation. It has been found that the numerical results match well with the analytical solution when we take some appropriate parameters, which demonstrates that the current general propagation lattice Boltzmann model is a satisfactory and efficient method, and could be more stable and accurate than the standard lattice Bhatnagar-Gross-Krook model.

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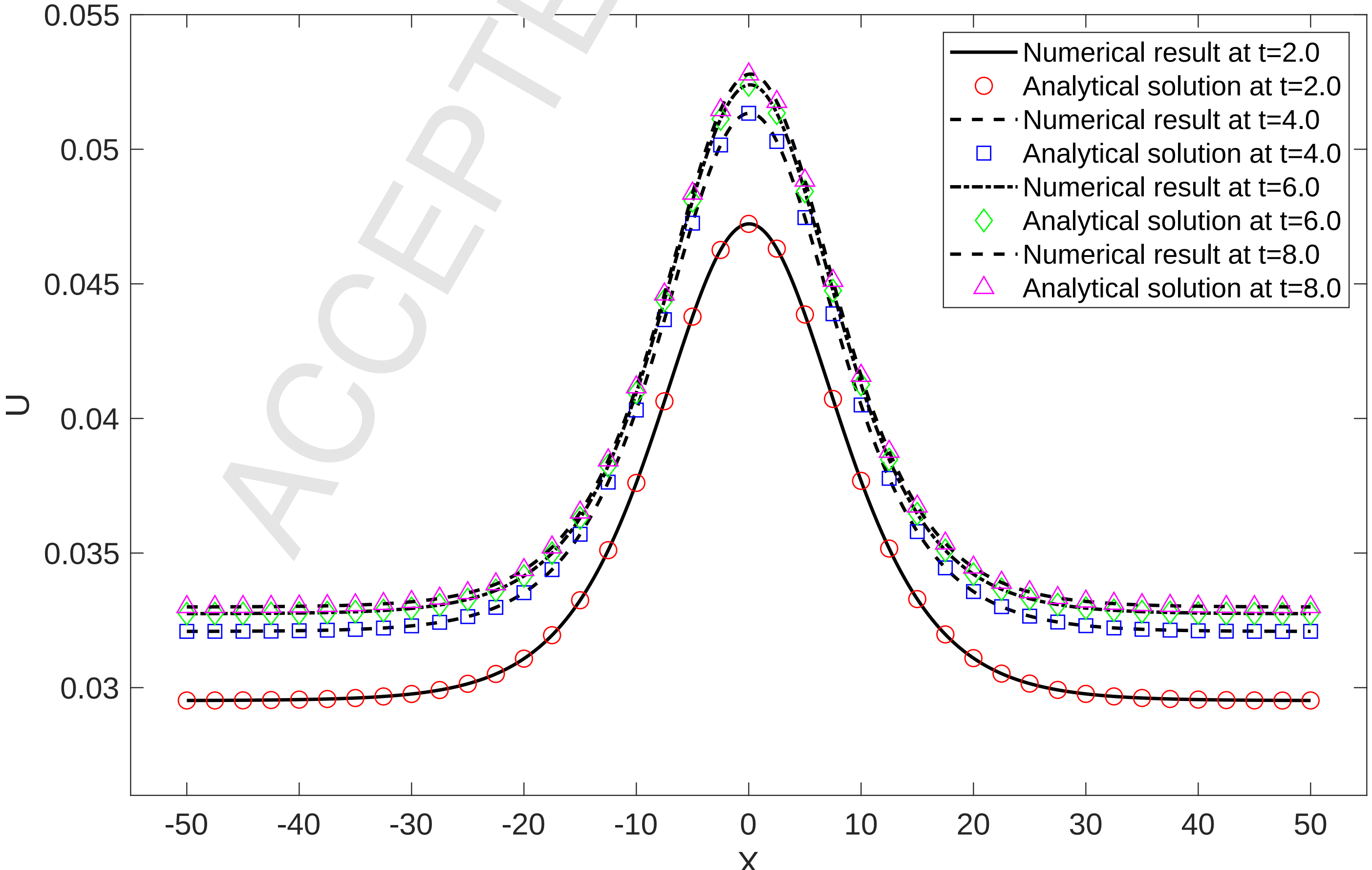
Figure



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Numerical result at t=2.0
Analytical solution at t=2.0
Numerical result at t=4.0
Analytical solution at t=4.0
Numerical result at t=6.0
Analytical solution at t=6.0
Numerical result at t=8.0
Analytical solution at t=8.0