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# Remarks on global well-posedness of mild solutions to the three-dimensional Boussinesq equations

Jiaqi Yang

*Key Laboratory for Mechanics in Fluid Solid Coupling Systems, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China*

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ABSTRACT

Recently, by using the argument of Lei & Lin (2011) [11], Liu & Gao (2017) [13] establish the global well-posedness of mild solutions to the three-dimensional Boussinesq equations in the space $\chi^{-1}$ defined by $\chi^{-1} = \{u \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{-1}|\hat{u}(\xi)|d\xi < \infty\}$. However, it seems that their proof is incorrect, and has some obvious and essential mistakes. Compared with the Navier-Stokes equations, it is difficulty to obtain a global well-posedness of mild solutions to the Boussinesq system in the space $\chi^{-1}$. In this paper, we will point out the mistakes of Liu & Gao. And, furthermore, in order to understand the difficulty of the Boussinesq system better, we study an illuminating system as follows:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu(1+t)^\alpha \Delta u + \nabla p = \theta e_3, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \partial_t\theta + (u\cdot\nabla)\theta - k(1+t)^\alpha\Delta\theta = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ u(x,0)=u^0, \quad \theta(x,0)=\theta^0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$, $k > 0$ and $\alpha > 1$ are real constant parameters. By using the time-weighted estimate, we can show that the above system has a global mild solution.


## 1. Introduction

Consider the three-dimensional Boussinesq system,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \mu\Delta u + \nabla p = \theta e_3, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \partial_t\theta + (u\cdot\nabla)\theta - k\Delta\theta = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^3\times(0,\infty),\\ u(x,0)=u^0, \quad \theta(x,0)=\theta^0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

*E-mail addresses:* yjqmath@163.com, yjq@imech.ac.cn.

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where $u = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denotes the velocity, $p = p(x,t) \in \mathbb{R}$ denotes scalar pressure, and $\theta \in \mathbb{R}$ is the temperature, $\mu > 0$ is the constant kinematic viscosity, $k > 0$ is the thermal diffusivity, $e_3 = (0,0,1)^T$, while $u^0$ and $\theta^0$ are the given initial velocity and initial temperature, respectively.

When $\theta \equiv 0$, the system (1) becomes the classical Navier-Stokes equations. The global existence of weak solutions is obtained by Leray [12]. Fujita & Kato [6], Kato [9], Koch & Tataru [10] proved that the global well-posedness of strong solutions for small initial data in the space $\dot{H}^s$, $s \geq \frac{1}{2}$, the Lebesgue space $L^3(\mathbb{R}^3)$, the space $\mathrm{BMO}^{-1}$, respectively. Recently, Lei & Lin [11] gave global mild solutions of Navier-Stokes equations in $C(\mathbb{R}^+, \chi^{-1}) \cap L^1(\mathbb{R}^+, \chi^1)$ with initial data $\|u^0\|_{\chi^{-1}} < \mu$, where

$$\chi^s = \{u \in \mathcal{D}'(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi)| d\xi < \infty\}$$

with the norm

$$\|u\|_{\chi^s} = \int\limits_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi)| d\xi.$$

The Boussinesq system arises from the density dependent incompressible Navier-Stokes equations by using the Boussinesq approximation, which consists in neglecting the density dependence in all the terms but the one involving the gravity. It has important roles in the atmospheric sciences [14], as well as being a model in many geophysical applications [15]. Due to its connection to incompressible Navier-Stokes equations, the Boussinesq system had lately received significant attention in mathematical fluid dynamics, see, e.g., [1–5,7,8,16]. By using the argument of Lei & Lin [11], recently, Liu & Gao [13] established a global well-posedness of mild solutions to 3D Boussinesq equations. However, their proof has some obvious and essential mistakes as follows:

- On page 8654 of [13], by, see (3.4) in [13],

$$\begin{cases} \partial_t \hat{u} + i\xi \cdot \int \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) d\eta + \mu |\xi|^2 \hat{u} + i\xi \hat{p} - \hat{\theta} e_3 = 0, \\ \partial_t \hat{\theta} + i\xi \cdot \int \hat{u}(\eta) \otimes \hat{\theta}(\xi - \eta) d\eta + k|\xi|^2 \hat{\theta} = 0, \\ \xi \cdot \hat{u} = 0, \end{cases} \tag{2}$$

  the authors obtained the following inequality, see (3.5) in [13],

$$\begin{aligned} &\frac{d}{dt} \int \left( |\xi|^{-1} |\hat{u}| + |\xi|^{-1} |\hat{\theta}| \right) d\xi + \mu \int |\xi| |\hat{u}| d\xi + k \int |\xi| |\hat{\theta}| d\xi \boxed{+ \int |\xi|^{-1} |\hat{\theta}| d\xi} \\ &\leq \iint |\hat{u}(\eta)| |\hat{u}(\xi - \eta)| d\eta d\xi + \iint |\hat{u}(\eta)| |\hat{\theta}(\xi - \eta)| d\eta d\xi. \end{aligned}$$

  This is obviously incorrect, since one can not ensure the sign in front of the integral $\int |\xi|^{-1} |\hat{\theta}| d\xi$ is positive. Actually, by (2), one has

$$\begin{aligned} \partial_t |\hat{u}| =& \frac{1}{2|\hat{u}|} \left( \bar{\hat{u}} \cdot \partial_t \hat{u} + \hat{u} \cdot \partial_t \bar{\hat{u}} \right) \\ =& -\mu |\xi|^2 |\hat{u}| + \left( \frac{\overline{\hat{u}_3}}{2|\hat{u}|} \hat{\theta} + \frac{\hat{u}_3}{2|\hat{u}|} \bar{\hat{\theta}} \right) \\ & - \frac{i}{2|\hat{u}|} \xi \cdot \left( \int \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) d\eta \cdot \overline{\hat{u}(\xi)} - \int \overline{\hat{u}(\eta)} \otimes \overline{\hat{u}(\xi - \eta)} d\eta \cdot \hat{u}(\xi) \right), \end{aligned} \tag{3}$$

where we have used (2)$_3$ to cancel the term including the pressure and

$$\begin{aligned}\partial_t|\hat{\theta}| =&\frac{1}{2|\hat{\theta}|}\left(\bar{\hat{\theta}}\partial_t\hat{\theta}+\hat{\theta}\partial_t\bar{\hat{\theta}}\right)\\ =&-k|\xi|^2|\hat{\theta}|\\ &-\frac{i}{2|\hat{\theta}|}\xi\cdot\left(\int\hat{u}(\eta)\otimes\hat{\theta}(\xi-\eta)d\eta\cdot\overline{\hat{\theta}(\xi)}-\int\overline{\hat{u}(\eta)}\otimes\overline{\hat{\theta}(\xi-\eta)}d\eta\cdot\hat{\theta}(\xi)\right).\end{aligned}\tag{4}$$

One can only get

$$\begin{aligned}&\frac{d}{dt}\int\left(|\xi|^{-1}|\hat{u}|+|\xi|^{-1}|\hat{\theta}|\right)d\xi+\mu\int|\xi||\hat{u}|d\xi+k\int|\xi||\hat{\theta}|d\xi\boxed{-\int|\xi|^{-1}|\hat{\theta}|d\xi}\\ &\le\iint|\hat{u}(\eta)||\hat{u}(\xi-\eta)|d\eta d\xi+\iint|\hat{u}(\eta)||\hat{\theta}(\xi-\eta)|d\eta d\xi.\end{aligned}$$

However, the term $-\int|\xi|^{-1}|\hat{\theta}|d\xi$ is a bad term, one must cancel it, otherwise one can not obtain the global well-posedness.

- On pages 8660-8661 of [13], from, see (4.4) in [13],

$$\begin{aligned}&\|\omega_m(t)\|_{\chi^{-1}}+\|\vartheta_m(t)\|_{\chi^{-1}}+\frac{\mu}{2}\int_0^t\|\omega_m(t)\|_{\chi^1}dt+\frac{k}{2}\int_0^t\|\vartheta_m(t)\|_{\chi^1}dt\\ &\le\|\omega_m^0(t)\|_{\chi^{-1}}+\|\vartheta_m^0(t)\|_{\chi^{-1}}\end{aligned}$$

and, see (4.14) in [13],

$$\begin{aligned}&\|A_m(t)\|^2+\|B_m(t)\|^2+\mu\int\|\nabla A_m(\tau)\|^2\tau+k\int_0^t\|\nabla B_m(\tau)\|^2d\tau\\ &\le\left(\|A_m^0\|^2+\|B_m^0\|^2\right)e^{C\int_0^t\left(1+\|\omega_m(\tau)\|_{\chi^{-1}}\|\omega_m(\tau)\|_{\chi^1}+\|\vartheta_m(\tau)\|_{\chi^{-1}}\|\vartheta_m(\tau)\|_{\chi^1}\right)d\tau},\end{aligned}$$

the authors obtained the following inequality, see (4.15) in [13],

$$\begin{aligned}&\|A_m(t)\|^2+\|B_m(t)\|^2+\mu\int\|\nabla A_m(\tau)\|^2\tau+k\int_0^t\|\nabla B_m(\tau)\|^2d\tau\\ &\le\left(\|A_m^0\|^2+\|B_m^0\|^2\right)\boxed{e^{C\left(1+\|\omega_m^0\|^2_{\chi^{-1}}+\|\vartheta_m^0\|^2_{\chi^{-1}}\right))}}.\end{aligned}$$

This is obviously false, actually, one can only obtain

$$\begin{aligned}&\|A_m(t)\|^2+\|B_m(t)\|^2+\mu\int\|\nabla A_m(\tau)\|^2\tau+k\int_0^t\|\nabla B_m(\tau)\|^2d\tau\\ &\le\left(\|A_m^0\|^2+\|B_m^0\|^2\right)\boxed{e^{Ct}e^{C\left(\|\omega_m^0\|^2_{\chi^{-1}}+\|\vartheta_m^0\|^2_{\chi^{-1}}\right)}}.\end{aligned}$$

Thus, one can not obtain a decay result like [13].

From the above analyses, it is difficulty to establish a global well-posedness of the three-dimensional Boussinesq equations in the space $\chi^{-1}$. Maybe one can not obtain a global well-posedness in the space $\chi^{-1}$. However, it is interesting that, by using the time-weighted estimate, we can prove the global well-posedness of the following generalized Boussinesq system with time dependent viscosity coefficients in the space $\chi^{-1}$,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu(t)\Delta u + \nabla p = \theta e_3, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \partial_t \theta + (u \cdot \nabla)\theta - k(t)\Delta\theta = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u^0, \quad \theta(x,0) = \theta^0, & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

with $\mu(t) = \mu(1+t)^\alpha$, $k(t) = k(1+t)^\alpha$, and $\mu > 0$, $k > 0$ and $\alpha > 1$ are real constant parameters. Specifically, we have the following Theorem.

**Theorem 1.** *Suppose that $u^0$, $\theta^0 \in \chi^{-1}$ satisfying*

$$\|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}} < \min\{\mu, k\}, \tag{6}$$

*then the Boussinesq system (5) has a unique global solution*

$$\big((1+t)^{-\alpha}u, (1+t)^{1-\alpha}\theta\big) \in C([0,+\infty); \chi^{-1}),$$

*and*

$$(u, (1+t)\theta) \in L^1([0,+\infty); \chi^1).$$

*Moreover, for any $t \in [0,\infty)$, one has the following*

$$\begin{aligned} &(1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}} \\ &+ \left(\mu - \|u^0\|_{\chi^{-1}} - \frac{\alpha}{2(\alpha-1)}\|\theta^0\|_{\chi^{-1}}\right)\int_0^t \|u(s)\|_{\chi^1} ds \\ &+ \frac{\alpha}{\alpha-1}\left(k - \frac{1}{2}\|u^0\|_{\chi^{-1}}\right)\int_0^t (1+s)\|\theta(s)\|_{\chi^1} ds \\ &\le \|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}}. \end{aligned}$$

**Remark 1.** When $\alpha = 0$, the system (5) becomes the Boussinesq system. However, we have no idea to deal with the case $\alpha \le 1$. There is a big gap between $\alpha = 0$ and $\alpha = 1$. Hence, it is difficulty to obtain a global well-posedness of mild solutions to the Boussinesq system in the space $\chi^{-1}$.

## 2. The proof of Theorem 1

This section is devoted to the proof of the Theorem 1. The existence and uniqueness of local smooth solutions can be done as [11]; thus, we may assume that $(u, \theta)$ is smooth enough in the interval $[0, T)$. We will establish some a priori bounds that will allow us to show $T = \infty$ under the assumption (6). Taking the Fourier transform of the system (1), one can get (2), i.e.

$$\begin{cases} \partial_t \hat{u} + i\xi \cdot \int \hat{u}(\eta) \otimes \hat{u}(\xi - \eta) d\eta + \mu (1+t)^{\alpha} |\xi|^2 \hat{u} + i\xi \hat{p} - \hat{\theta} e_3 = 0, \\ \partial_t \hat{\theta} + i\xi \cdot \int \hat{u}(\eta) \otimes \hat{\theta}(\xi - \eta) d\eta + k(1+t)^{\alpha} |\xi|^2 \hat{\theta} = 0, \\ \xi \cdot \hat{u} = 0. \end{cases}$$

Multiplying (3) by $(1+t)^{-\alpha}|\xi|^{-1}$ and integrating with respect to $\xi$, one can get

$$\begin{aligned} &\frac{d}{dt}\int (1+t)^{-\alpha}|\xi|^{-1}|\hat{u}|d\xi + \frac{\alpha}{(1+t)^{1+\alpha}}\int |\xi|^{-1}|\hat{u}|d\xi + \mu \int |\xi||\hat{u}|d\xi \\ \le &\frac{\alpha}{(1+t)^{\alpha}}\int |\xi|^{-1}|\hat{\theta}|d\xi + \frac{1}{(1+t)^{\alpha}}\iint |\hat{u}(\eta)||\hat{u}(\xi-\eta)|d\eta d\xi \end{aligned} \tag{7}$$

Multiplying (4) by $\frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}|\xi|^{-1}$ and integrating with respect to $\xi$, one gets

$$\begin{aligned} &\frac{d}{dt}\int \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}|\xi|^{-1}|\hat{\theta}|d\xi + \frac{\alpha}{(1+t)^{\alpha}}\int |\xi|^{-1}|\hat{\theta}|d\xi + \frac{k\alpha}{\alpha-1}\int (1+t)|\xi||\hat{\theta}|d\xi \\ &\le \iint \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}|\hat{u}(\eta)||\hat{\theta}(\xi-\eta)|d\eta d\xi. \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned} &\frac{d}{dt}\int \left((1+t)^{-\alpha}|\xi|^{-1}|\hat{u}| + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}|\xi|^{-1}|\hat{\theta}|\right) d\xi + \frac{\alpha}{(1+t)^{1+\alpha}}\int |\xi|^{-1}|\hat{u}|d\xi \\ &\quad + \mu\int |\xi||\hat{u}|d\xi + \frac{k\alpha}{\alpha-1}\int (1+t)|\xi||\hat{\theta}|d\xi \\ &\le \frac{1}{(1+t)^{\alpha}}\iint |\hat{u}(\eta)||\hat{u}(\xi-\eta)|d\eta d\xi + \iint \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}|\hat{u}(\eta)||\hat{\theta}(\xi-\eta)|d\eta d\xi \\ &\le \frac{1}{2(1+t)^{\alpha}}\iint \left(|\eta|^{-1}|\xi-\eta| + |\eta||\xi-\eta|^{-1}\right)|\hat{u}(\eta)||\hat{u}(\xi-\eta)|d\eta d\xi \\ &\quad + \frac{\alpha}{2(\alpha-1)}(1+t)^{1-\alpha}\iint \left(|\eta|^{-1}|\xi-\eta| + |\eta||\xi-\eta|^{-1}\right)|\hat{u}(\eta)||\hat{\theta}(\xi-\eta)|d\eta d\xi \\ &\le \int (1+t)^{-\alpha}|\xi|^{-1}|\hat{u}|d\xi \int |\xi||\hat{u}|d\xi + \frac{\alpha}{2(\alpha-1)}\int (1+t)^{-\alpha}|\xi|^{-1}|\hat{u}|d\xi\int (1+t)|\xi||\hat{\theta}|d\xi \\ &\quad + \frac{\alpha}{2(\alpha-1)}\int |\xi||\hat{u}|d\xi\int (1+t)^{1-\alpha}|\xi|^{-1}|\hat{\theta}|d\xi \\ &\le \|(1+t)^{-\alpha}u\|_{\chi^{-1}}\|u\|_{\chi^1} + \frac{\alpha}{2(\alpha-1)}\|(1+t)^{-\alpha}u\|_{\chi^{-1}}\|(1+t)\theta\|_{\chi^1} \\ &\quad + \frac{\alpha}{2(\alpha-1)}\|(1+t)^{1-\alpha}\theta\|_{\chi^{-1}}\|u\|_{\chi^1}. \end{aligned} \tag{9}$$

From assumption (6), one has

$$(1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}} < \min\{\mu, k\}$$

for a very short time interval $[0,\delta]$ with $0<\delta<T$. Due to (9), we have for any $t\in[0,\delta]$

$$\frac{d}{dt}\left((1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}}\right) \le 0.$$

By a continuity argument in the time variable, one can get

$$(1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}} \le \|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}} < \min\{\mu, k\},$$

for any $t \in [0, T)$. Thus, for any $t \in [0, T)$

$$\begin{aligned}
&(1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}} \\
&+ \left(\mu - \|u^0\|_{\chi^{-1}} - \frac{\alpha}{2(\alpha-1)}\|\theta^0\|_{\chi^{-1}}\right)\int_0^t \|u(s)\|_{\chi^1} ds \\
&+ \frac{\alpha}{\alpha-1}\left(k - \frac{1}{2}\|u^0\|_{\chi^{-1}}\right)\int_0^t (1+s)\|\theta(s)\|_{\chi^1} ds \\
&\le \|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}}.
\end{aligned} \tag{10}$$

It is straightforward to show that

$$\|\nabla u\|_{L^\infty} \le \int |\xi||\hat{u}| d\xi \le \|u\|_{\chi^1}$$

and

$$\|\nabla \theta\|_{L^\infty} \le \int |\xi||\hat{\theta}| d\xi \le \|\theta\|_{\chi^1}.$$

Thus

$$\int_0^T \|\nabla u\|_{L^\infty} dt \le \frac{\|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}}}{\mu - \|u^0\|_{\chi^{-1}} - \frac{\alpha}{2(\alpha-1)}\|\theta^0\|_{\chi^{-1}}}$$

and

$$\int_0^T \|\nabla \theta\|_{L^\infty} dt \le \frac{\frac{\alpha}{\alpha-1}\|u^0\|_{\chi^{-1}} + \|\theta^0\|_{\chi^{-1}}}{k - \frac{1}{2}\|u^0\|_{\chi^{-1}}}.$$

By the standard energy method, we can show that for any $m > 0$

$$\begin{aligned}
&\|u(t)\|_{H^m} + \|\theta(t)\|_{H^m} \\
&\le \left(\|u^0\|_{H^m} + \|\theta^0\|_{H^m}\right) exp\left\{C_m \int_0^T \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty}\right) dt\right\} \\
&\le \left(\|u^0\|_{H^m} + \|\theta^0\|_{H^m}\right) e^{C_m T} \\
&\quad \times exp\left\{C_m\left(\frac{\|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}}}{\mu - \|u^0\|_{\chi^{-1}} - \frac{\alpha}{2(\alpha-1)}\|\theta^0\|_{\chi^{-1}}} + \frac{\frac{\alpha}{\alpha-1}\|u^0\|_{\chi^{-1}} + \|\theta^0\|_{\chi^{-1}}}{k - \frac{1}{2}\|u^0\|_{\chi^{-1}}}\right)\right\},
\end{aligned}$$

which implies that $T = \infty$. Hence, from (10), we have for any $t \in [0, \infty)$

$$\begin{aligned}
&(1+t)^{-\alpha}\|u(t)\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}(1+t)^{1-\alpha}\|\theta(t)\|_{\chi^{-1}} \\
&+ \left(\mu - \|u^0\|_{\chi^{-1}} - \frac{\alpha}{2(\alpha-1)}\|\theta^0\|_{\chi^{-1}}\right)\int_0^t \|u(s)\|_{\chi^1} ds \\
&+ \frac{\alpha}{\alpha-1}\left(k - \frac{1}{2}\|u^0\|_{\chi^{-1}}\right)\int_0^t (1+s)\|\theta(s)\|_{\chi^1} ds \\
\leq & \|u^0\|_{\chi^{-1}} + \frac{\alpha}{\alpha-1}\|\theta^0\|_{\chi^{-1}}.
\end{aligned}$$

This completes the proof of Theorem 1.

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