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PIMOL: The Finite Difference Method of Lines Based on the Precise Integration Method for an Arbitrary Irregular Domain

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ABSTRACT The aim of this paper is to introduce a new semi-analytical method named precise integration method of lines (PIMOL), which is developed and used to solve the ordinary differential equation (ODE) systems based on the finite difference method of lines and the precise integration method. The irregular domain problem is mainly discussed in this paper. Three classical examples of Poisson's equation problems are given, including one regular and two irregular domain examples. The PIMOL reduces a semi-discrete ODE problem to a linear algebraic matrix equation and does not require domain mapping for treating the irregular domain problem. Numerical results show that the PIMOL is a powerful method.

KEY WORDS PIMOL, ODEs, FDMOL, PIM, Poisson's equation problems, Semi-analytical (discrete), Irregular domain

1. Introduction

The newly developed semi-analytical algorithm scheme for solving boundary-value problems (BVPs) of elliptic type, i.e., the precise integration method of lines (PIMOL) discussed in this paper, is based on the finite difference method of lines (FDMOL) and the precise integration method (PIM). For a review of the method of line (MOL)-related studies, see, for instance, Ref. [1]. The key is to semi-discretize a partial differential equation (PDE) into a system of ordinary differential equations (ODEs) defined on discrete lines by replacing the derivatives with respect to all but independent variables with finite differences (FDs). The resulting two-point boundary-value ODEs may then be solved by analytical or numerical methods. The requirement of regular domain, inflexibility of meshes and ODE solving, or the conventional MOL did not attract much attention, and the related investigations and applications were limited. Some applications of MOL in the Poisson's equation and other BVPs can be found in Ref. [2–5] by Meyer and Janac, respectively. The MOL has also been applied to solid mechanics by Irob [6], Gyekenyesi and Mendelson [7], Malik and Fu [8, 9], Mendelson and Alam [10], and Alam and Mendelson [11]. In most of the aforementioned applications, the ODEs were solved by ad hoc shooting-like numerical processes. Jones et al. [12], however, studied the convergence of the MOL solution and found that the ODEs resulting from the MOL may be inherently unstable for the shooting methods. Xanthis [13, 14] and Yuan [15] solved the system of ODEs using an ODE solver and, furthermore,

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developed a new computational tool in structure analysis FEMOL based on the finite element discrete ideas and a modern ODE solver [15–17]. The PIM is a powerful method for solving ODEs of both initial problems and boundary-value problems [18–20]. Reference [21] developed a PIMOL scheme in the regular domain. In this paper, the PIMOL scheme in an arbitrary irregular domain is investigated. The old method FDMOL equipped with PIM will be gaining new value, power, and efficiency in the future.

2. PIMOL: Standard Formulation of FDMOL Based on PIM

2.1. The Finite Difference Method of Lines

To explore the FDMOL and the PIMOL, we consider the following Poisson's equation defined on a rectangular domain, as shown in Fig. 1 [17]:

$$\nabla^2 u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = -f \tag{1}$$

which is subject to the Dirichlet boundary conditions (BCs):

$$u = 0 \qquad (x = \pm a, \quad y = \pm b) \tag{2}$$

For simplicity, we assume that $f(x, y)$ is bi-symmetrical. Thus, we can solve the problem on a quarter of the domain, which is semi-discretized by $N + 1$ equally spaced vertical lines with distance $h = a/N$, as shown in Fig. 2.

By defining

$$u_i = u_i(y) = u(x_i, y) \tag{3}$$

and using the three-point central difference of accuracy $O(h^2)$ to approximate the partial derivatives with respect to the independent variable x at $x = x_i$,

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2); \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \tag{4}$$

The typical FDMOL equation on an interior line i can be written as a second-order ODE of the following:

$$u_i'' = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - f_i \qquad (y \in (0, b), \quad i = 2, 3, \ldots, N) \tag{5}$$

where $f_i = f_i(y) = f(x_i, y)$. To establish the FDMOL equation on the first line, an auxiliary line $i = 0$ is introduced. Using the Neumann-type BC, $\partial u/\partial x = 0$ at the left boundary line yields $u_0 = u_2$, which eliminates the line function u_0 at the auxiliary line. The ODE of the first line can be rewritten as

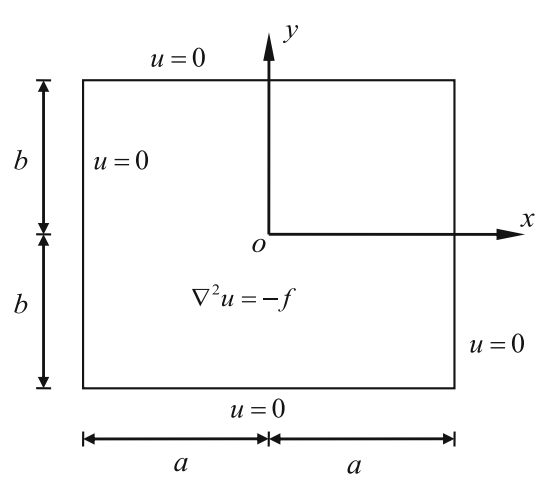


Fig. 1. The Poisson's equation

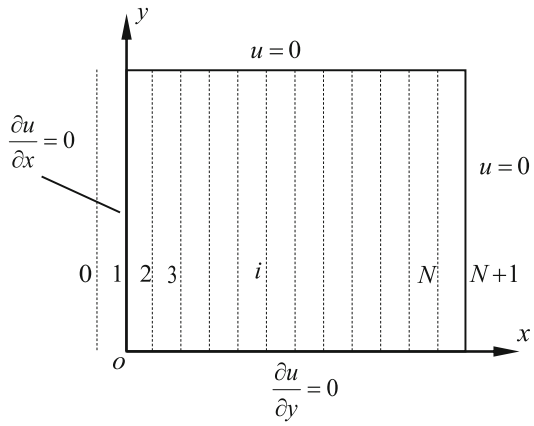


Fig. 2. PIMOL mesh for a quarter domain

$$u_1'' = -\frac{2u_2 - 2u_1}{h^2} - f_1 \qquad (y \in (0, b)) \tag{6}$$

For the right boundary line $i = N+1$, since $u_{N+1} = 0$, the ODE on line $i = N$ can be rewritten as

$$u_N'' = -\frac{-2u_N + u_{N-1}}{h^2} - f_N \qquad (y \in (0, b)) \tag{7}$$

The end-point BCs for each line are

$$u_i'(0) = 0, \qquad u_i(b) = 0 \qquad (i = 1, 2, 3, \ldots, N) \tag{8}$$

2.2. Precise Integration Method [18]

An ordinary differential equation of any order can be always changed into an equivalent system of first-order ODEs. A set of ODEs can be given in the matrix/vector form as

$$\mathbf{v}' = \mathbf{A}\mathbf{v} + \mathbf{f} \tag{9}$$

where a prime ($'$) stands for the derivative with respect to ξ, $\mathbf{v}(\xi)$ is an n-dimensional vector function to be determined, $\mathbf{A}$ is a given $n \times n$ constant matrix, and $\mathbf{f}(\xi)$ is a given n-dimensional external force vector.

For the homogeneous equations,

$$\mathbf{v}' = \mathbf{A}\mathbf{v} \tag{10}$$

because $\mathbf{A}$ is a ξ-invariant matrix, its general solution can be given as

$$\mathbf{v} = \exp(\mathbf{A}\xi) \cdot \mathbf{v}_0 \tag{11}$$

where $\mathbf{v}_0 = \mathbf{v}(\xi_0)$ is assumed to be a known vector boundary condition.

The solution of Eq. (9) can be obtain using Duhamel's integration

$$\mathbf{v} = \exp(\mathbf{A} \cdot (\xi - \xi_0)) \cdot \mathbf{v}_0 + \int_{\xi_0}^{\xi} \exp(\mathbf{A} \cdot (\xi - \zeta))\, \mathbf{f}(\zeta)\, \mathrm{d}\zeta \tag{12}$$

The focuses are on the precise calculation of $\exp(\mathbf{A}t)$, $t = \xi - \xi_0$ for a given ξ and the precise numerical calculation of the second integration part, and the precise numerical integration also focuses on the precise computation of $\exp(\mathbf{A}t)$ for a given $t = \xi - \zeta$.

The exponential matrix is defined as usual

$$\mathbf{T} = \exp(\mathbf{A}\xi) = \mathbf{I} + \mathbf{A}\xi + \frac{(\mathbf{A}\xi)^2}{2} + \frac{(\mathbf{A}\xi)^3}{3!} + \frac{(\mathbf{A}\xi)^4}{4!} + \cdots \tag{13}$$

the focus of which is on the numerical calculation of the exponential matrix $\mathbf{T}$, as precise as possible.

(1) Direct Computation Method

Sufficient terms of the series are taken to guarantee the precise of matrix $\mathbf{T}$. For a large N,

$$\mathbf{T} \approx \mathbf{T}_N = \frac{(\mathbf{A}\xi)}{N} \cdot \frac{(\mathbf{A}\xi)^{(N-1)}}{(N-1)!} + \mathbf{T}_{N-1} \tag{14}$$

(2) The 2N Algorithm

Using the additional theorem of the exponential function,

$$\mathbf{T} = \exp(\mathbf{A}\xi) \equiv \left[\exp\left(\mathbf{A}\frac{\xi}{2^N}\right)\right]^{2^N} = [\exp(\mathbf{A}\zeta)]^{2^N}, \quad \zeta = \frac{\xi}{2^N} \tag{15}$$

$$\exp(\mathbf{A}\zeta) \approx \mathbf{I} + \mathbf{A}\zeta + \frac{(\mathbf{A}\zeta)^2}{2} + \frac{(\mathbf{A}\zeta)^3}{3!} + \frac{(\mathbf{A}\zeta)^4}{4!} + \cdots + \frac{(\mathbf{A}\zeta)^K}{K!} = \mathbf{I} + \mathbf{T_a} \tag{16}$$

For matrix $\mathbf{T}$, Eq. (15) should be factored into

$$\mathbf{T} = (\mathbf{I} + \mathbf{T}_a)^{2^N} = [(\mathbf{I} + \mathbf{T}_a) \cdot (\mathbf{I} + \mathbf{T}_a)]^{2^{N-1}} = \left[\mathbf{I} + \left(2\mathbf{T}_a + (\mathbf{T}_a)^2\right)\right]^{2^{N-1}} \tag{17}$$

Such factorization should be iterated N times

$$\mathbf{T}_a = 2\mathbf{T}_a + (\mathbf{T}_a)^2, \quad N \text{ times}, \tag{18}$$

and the summation

$$\mathbf{T} = \mathbf{I} + \mathbf{T}_a \tag{19}$$

is finally executed.

2.3. PIMOL Algorithm

In order to change the governing equations of Eqs. (5)–(7) into an equivalent system of first-order ODEs, we define a new identity function on each line as

$$v_i = u_i' \qquad (y \in (0, b), \quad i = 1, 2, 3, \ldots, N) \tag{20}$$

and then the governing equations of Eqs. (5)–(7) can be rewritten as the following equivalent system of first-order ODEs:

$$\begin{cases} v_1' = -\frac{2u_2 - 2u_1}{h^2} - f_1 \\ v_i' = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - f_i \\ v_N' = -\frac{-2u_N + u_{N-1}}{h^2} - f_N \end{cases} \qquad (y \in (0, b), \quad i = 2, 3, \ldots, N) \tag{21}$$

Based on Eqs. (20) and (21), a set of first-order ODEs can be given in the matrix/vector form as

$$\mathbf{U}' = \mathbf{AU} + \mathbf{F} \qquad (y \in (0, b)) \tag{22}$$

where $\mathbf{A}$ is a $2N \times 2N$ matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{a} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{a} = \frac{1}{h^2} \begin{array}{c} \begin{bmatrix} 2 & -2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix} \\ i \end{array} \quad i$$

$a_{11} = 2, \quad a_{12} = -2$

$a_{i\,i-1} = -1, \quad a_{ii} = 2, \quad a_{i\,i+1} = -1, \quad i = 2, 3, 4, \ldots, N$

the others: $a_{ij} = 0$

$$\mathbf{U} = [v_1, v_2, v_3, \ldots v_i, \ldots, v_N, u_1, u_2, u_3, \ldots u_i, \ldots u_N]^{\mathrm{T}}$$

$$\mathbf{F} = [-f_1, -f_2, -f_3, \ldots, -f_i, \ldots, -f_N, 0, 0, 0, \ldots, 0 \ldots, 0]^{\mathrm{T}}$$

In addition, the end-point BCs for each line can also be rewritten as

$$v_i(0) = 0, \qquad u_i(b) = 0 \qquad (i = 1, 2, 3, \ldots, N) \tag{23}$$

2.4. Solution Algorithm

The solutions of Eq. (22) can be expressed as the following by using Duhamel's integration as Eq. (12):

$$\mathbf{U}(y) = \exp(\mathbf{A}y)\mathbf{U}_0 + \int_0^y \exp(\mathbf{A} \cdot (y - t))\, \mathbf{F}(t)\, \mathrm{d}t \qquad (y \in (0, b)) \tag{24}$$

When $y = b$, we have

$$\mathbf{U}_b = \mathbf{T}_b \mathbf{U}_0 + \hat{\mathbf{F}}_b, \quad \mathbf{T}_b = \exp(\mathbf{A}b), \quad \hat{\mathbf{F}}_b = \int_0^b \exp(\mathbf{A} \cdot (b - t))\, \mathbf{F}(t)\, \mathrm{d}t \tag{25}$$

where $\mathbf{T}_b$ is a $2N \times 2N$ matrix and $\hat{\mathbf{F}}_b$ is a $2N$ vector.

Substituting Eq. (23) of the end-point BCs into Eq. (25) yields

$$\begin{aligned}
&\mathbf{U}_b = \mathbf{T}_b \mathbf{U}_0 + \hat{\mathbf{F}}_b \\
&\mathbf{U}_0 = [0, 0, 0, \ldots 0, u_1(0), u_2(0), u_3(0), \ldots u_i(0), \ldots u_N(0)]^{\mathrm{T}} = [\mathbf{0} \quad \bar{\mathbf{u}}_0]^{\mathrm{T}} \\
&\mathbf{U}_b = [v_1(b), v_2(b), v_3(b), \ldots v_i(b), \ldots, v_N(b), 0, 0, 0, \ldots 0]^{\mathrm{T}} = [\bar{\mathbf{v}}_b \quad \mathbf{0}]^{\mathrm{T}} \\
&\mathbf{T}_b = \begin{bmatrix} \bar{\mathbf{T}}_{11} & \bar{\mathbf{T}}_{12} \\ \bar{\mathbf{T}}_{21} & \bar{\mathbf{T}}_{22} \end{bmatrix}, \quad \hat{\mathbf{F}}_b = \begin{bmatrix} \bar{\mathbf{F}}_1 & \bar{\mathbf{F}}_2 \end{bmatrix}^{\mathrm{T}}
\end{aligned} \tag{26}$$

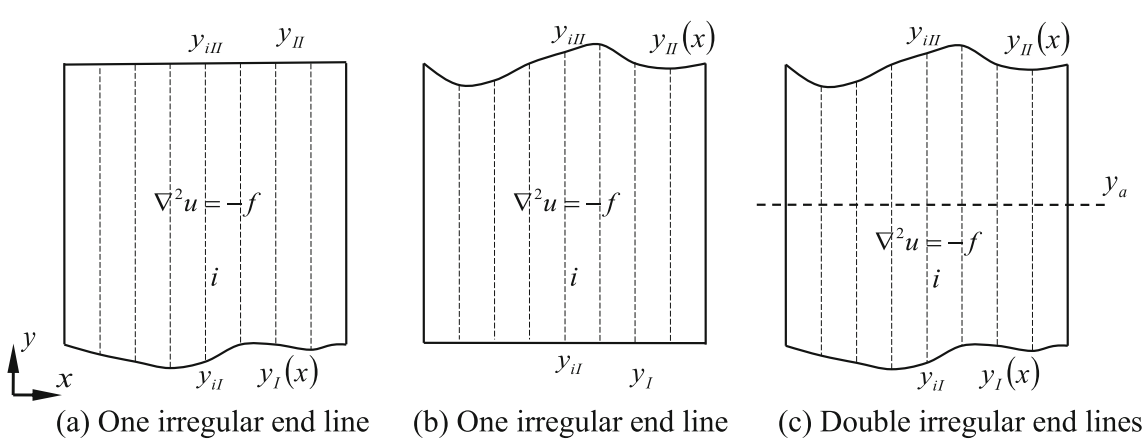
Notice that a semi-discrete BV ODE problem is reduced to a set of linear algebraic equations with $2N$ unknowns. It is easy to obtain that

$$\begin{cases} \bar{\mathbf{u}}_0^{\mathrm{T}} = -\bar{\mathbf{T}}_{22}^{-1} \bar{\mathbf{F}}_2^{\mathrm{T}} \\ \bar{\mathbf{v}}_b^{\mathrm{T}} = \bar{\mathbf{T}}_{12} \bar{\mathbf{u}}_0^{\mathrm{T}} + \bar{\mathbf{F}}_1^{\mathrm{T}} \end{cases} \tag{27}$$

At this point, we can say that the problem is solved. For any point (x, y) in the domain in Fig. 2, the semi-analytic solutions with respect to y on each line can be obtained by using Eq. (24); for any x falling out of the mesh lines, the interpolation method and many other methods can be used to obtain a relatively accurate solution.

3. PIMOL: "Semi"-irregular Domain

Because most engineering problems are not always in the regular domain shown in Fig. 1, we discuss those that we call "semi-irregular domains," as shown in Fig. 3. One of the characteristics of these



(a) One irregular end line (b) One irregular end line (c) Double irregular end lines

Fig. 3. PIMOL mesh for semi-irregular domain

domains is two parallel boundary lines. We arrange equally spaced vertical lines parallel to the parallel boundary lines for the PIMOL meshing. Only two cases need to be studied: one with one irregular end line and the other with double irregular end lines.

3.1. The Domain with One Irregular End Line

It suffices to discuss the case as shown in Fig. 3b because a is equivalent to b. The straight end line is perpendicular to the parallel boundary lines, and it is only a Neumann boundary condition or a Dirichlet boundary condition. For other cases, the straight end line is a Neumann BC and a Dirichlet BC, and an initial problem can be solved directly.

We here discuss the BCs in Fig. 3b:

$$\text{left}: \frac{\partial u}{\partial x}=0; \quad \text{right}: u=0; \quad \text{up}: \frac{\partial u}{\partial n}=0; \quad \text{bottom}: \frac{\partial u}{\partial y}=0$$

And other cases can be treated similarly. The solution can be expressed as Eq. (24):

$$\mathbf{U}(y)=\exp(\mathbf{A}\cdot(y-y_I))\mathbf{U}_{y_I}+\int_{y_I}^{y}\exp(\mathbf{A}\cdot(y-t))\,\mathbf{F}(t)\,\mathrm{d}t \tag{28}$$

When $y=y_{iII}$, we have

$$\mathbf{U}_{iII}=\mathbf{T}_{iII}\mathbf{U}_{y_I}+\hat{\mathbf{F}}_{iII}, \quad \mathbf{T}_{iII}=\exp(\mathbf{A}\cdot(y_{iII}-y_I)), \quad \hat{\mathbf{F}}_{iII}=\int_{y_I}^{y_{iII}}\exp(\mathbf{A}\cdot(y_{iII}-t))\,\mathbf{F}(t)\,\mathrm{d}t$$
$$(i=1,2,\ldots NL) \tag{29}$$

where for each line i, $\mathbf{T}_{iII}$ is a $2N\times 2N$ determined value matrix by PIM, and $\hat{\mathbf{F}}_{iII}$ is a $2N$ determined column vector by PIM and numerical integration.

The BCs at the lower end points are straight forward:

$$v_{iI}=0, \quad y=y_I \quad (i=1,2,3,\ldots,N) \tag{30}$$

Then, we have

$$\mathbf{U}_{y_I}=[0,0,\ldots,0,u_1(y_I),u_2(y_I),\ldots,u_{NL}(y_I)]^{\mathrm{T}} \tag{31}$$

and the BCs at the upper end point of each line can be approximated by replacing $\partial u/\partial x$ with the finite difference (FD) formula as follows:

$$\begin{aligned}
&v_{1II}=0 && (a)\\
&\frac{u_{i+1}(y_{iII})-u_{i-1}(y_{iII})}{2h}n_x\Big|_{iII}+v_{iII}\,n_y|_{iII}=0 \quad (y=y_{iII},\ i=2,3,\ldots,NL-1) && (b)\\
&\frac{-u_{i-1}(y_{iII})}{2h}n_x\Big|_{iII}+v_{iII}\,n_y|_{iII}=0 \quad (y=y_{iII}, \quad i=NL) && (c)
\end{aligned} \tag{32}$$

Substituting Eq. (29) into Eq. (32) yields

$$\begin{aligned}
&[T_{1II}[1,1],T_{1II}[1,2],\ldots,T_{1II}[1,2NL]]\mathbf{U}_{y_I}+\hat{\mathbf{F}}_{iII}[1]=0\\
&\left(\bar{\mathbf{T}}_{NL+i}+2h\frac{n_{yi}}{n_{xi}}\bar{\mathbf{T}}_i\right)\mathbf{U}_{y_I}+2h\frac{n_{yi}}{n_{xi}}\hat{\mathbf{F}}_{iII}[i]+\hat{\mathbf{F}}_{iII}[NL+i+1]\\
&\quad -\hat{\mathbf{F}}_{iII}[NL+i-1]=0 \quad (i=2,3,\ldots,NL)\\
&\bar{\mathbf{T}}_{NL+i}=[T_{iII}[NL+i+1,1]-T_{iII}[NL+i-1,1],\ldots,T_{iII}[NL+i+1,NL]\\
&\quad -T_{iII}[NL+i-1,2NL]]\\
&\quad\quad (i=2,3,\ldots,NL-1)\\
&\bar{\mathbf{T}}_{NL+i}=[-T_{iII}[2NL-1,1],-T_{iII}[2NL-1,2],\ldots,-T_{iII}[2NL-1,2NL]]. \qquad (i=NL)
\end{aligned} \tag{33}$$

Rearranging Eq. (32) gives

$$\begin{aligned}
&\tilde{\mathbf{T}}\mathbf{u}_{y_I}+\tilde{\mathbf{F}}=0\\
&\mathbf{u}_{y_I}=[u_1(y_I),u_2(y_I),\ldots,u_{NL}(y_I)]^{\mathrm{T}}=-\tilde{\mathbf{T}}^{-1}\tilde{\mathbf{F}}
\end{aligned} \tag{34}$$

The problem can be considered completely solved now. For any point (x, y) in the domain in Fig. 3b, the semi-analytic solutions with respect to y on each line can be obtained by using Eq. (28), and for any x falling out of the mesh lines, the interpolation method or many other methods can be used to obtain a relatively accurate solution.

3.2. The Domain with Double Irregular End Lines

We discuss the BCs in Fig. 3c:

$$\text{left: } \frac{\partial u}{\partial x} = 0; \quad \text{right: } u = 0; \quad \text{up: } \frac{\partial u}{\partial n} = 0; \quad \text{bottom: } \frac{\partial u}{\partial n} = 0$$

as an example, and other cases can be treated similarly. By introducing an auxiliary line $y = y_a$, the solution based on that line can be expressed as Eq. (28):

$$\mathbf{U}(y) = \exp(\mathbf{A}y)\mathbf{U}_{y_a} + \int_{y_a}^{y} \exp(\mathbf{A}\cdot(y-t))\,\mathbf{F}(t)\,\mathrm{d}t \tag{35}$$

When $y = y_{iI}$ and $y = y_{iII}$, we have

$$\begin{aligned}
&\mathbf{U}_{iI} = \mathbf{T}_{iI}\mathbf{U}_{y_a} + \hat{\mathbf{F}}_{ia}, \quad \mathbf{T}_{iI} = \exp(\mathbf{A}\cdot(y_{iI} - y_a)), \quad \hat{\mathbf{F}}_{iI} = \int_{y_a}^{y_{iI}} \exp(\mathbf{A}\cdot(y_{iI} - t))\,\mathbf{F}(t)\,\mathrm{d}t \\
&\mathbf{U}_{iII} = \mathbf{T}_{iII}\mathbf{U}_{y_a} + \hat{\mathbf{F}}_{ia}, \quad \mathbf{T}_{iII} = \exp(\mathbf{A}\cdot(y_{iII} - y_a)), \quad \hat{\mathbf{F}}_{iII} = \int_{y_I}^{y_{iII}} \exp(\mathbf{A}\cdot(y_{iII} - t))\,\mathbf{F}(t)\,\mathrm{d}t \\
&\qquad (i = 1, 2, \ldots NL)
\end{aligned} \tag{36}$$

where for each line $i, \mathbf{T}_{iI}$ and $\mathbf{T}_{iII}$ are $2N \times 2N$ determined value matrix by PIM, and $\hat{\mathbf{F}}_{iI}$ and $\hat{\mathbf{F}}_{iII}$ are $2N$ determined column vector by PIM and numerical integration.

The BCs at the lower and upper end points of each line can be approximated by replacing $\partial u/\partial x$ with the FD formula as follows:

$$\begin{aligned}
&v_{1I} = 0 && (a) \\
&\frac{u_{i+1}(y_{iI}) - u_{i-1}(y_{iI})}{2h} n_x\Big|_{iI} + v_{iI}\, n_y|_{iI} = 0 \quad (y = y_{iI},\ i = 2, 3, \ldots, NL-1) && (b) \\
&\frac{-u_{i-1}(y_{iI})}{2h} n_x\Big|_{iI} + v_{iI}\, n_y|_{iI} = 0 \quad (y = y_{iI},\ i = NL) && (c)
\end{aligned} \tag{37}$$

$$\begin{aligned}
&v_{1II} = 0 && (a) \\
&\frac{u_{i+1}(y_{iII}) - u_{i-1}(y_{iII})}{2h} n_x\Big|_{iII} + v_{iII}\, n_y|_{iII} = 0 \quad (y = y_{iII},\ i = 2, 3, \ldots, NL-1) && (b) \\
&\frac{-u_{i-1}(y_{iII})}{2h} n_x\Big|_{iII} + v_{iII}\, n_y|_{iII} = 0 \quad (y = y_{iII},\ i = NL) && (c)
\end{aligned} \tag{38}$$

Substituting Eq. (36) into Eqs. (37) and (38) yields

$$\begin{aligned}
(a)\quad &\begin{cases}
[T_{1I}[1,1], T_{1I}[1,2], \ldots, T_{1I}[1,2NL]]\mathbf{U}_{y_I} + \hat{\mathbf{F}}_{iI}[1] = 0 \\
\left(\hat{\mathbf{T}}_i + 2h\frac{n_{yiI}}{n_{xiI}}\hat{\mathbf{T}}_i\right)\mathbf{U}_{y_a} + 2h\frac{n_{yiI}}{n_{xiI}}\hat{\mathbf{F}}_{iI}[i] + \hat{\mathbf{F}}_{iI}[NL+i+1] - \hat{\mathbf{F}}_{iI}[NL+i-1] = 0 \\
\qquad\qquad (i = 2, 3, \ldots, NL) \\
\hat{\mathbf{T}}_1 = [T_{1I}[1,1], T_{1I}[1,2], \ldots, T_{1I}[1,2NL]] \\
\hat{\mathbf{T}}_i = [T_{iI}[NL+i+1,1] - T_{iI}[NL+i-1,1], \ldots, T_{iI}[NL+i+1,NL] - T_{iI}[NL+i-1,2NL]] \\
\qquad\qquad (i = 2, 3, \ldots, NL-1) \\
\hat{\mathbf{T}}_{NL} = [-T_{iI}[2NL-1,1], -T_{iI}[2NL-1,2], \ldots, -T_{iI}[2NL-1,2NL]] \qquad (i = NL)
\end{cases} \\
(b)\quad &\begin{cases}
[T_{iII}[1,1], T_{iII}[1,2], \ldots, T_{iII}[1,2NL]]\mathbf{U}_{y_a} + \hat{\mathbf{F}}_{iII}[1] = 0 \\
\left(\bar{\mathbf{T}}_{NL+i} + 2h\frac{n_{yiII}}{n_{xiII}}\bar{\mathbf{T}}_i\right)\mathbf{U}_{y_I} + 2h\frac{n_{yiII}}{n_{xiII}}\hat{\mathbf{F}}_{iII}[i] + \hat{\mathbf{F}}_{iII}[NL+i+1] - \hat{\mathbf{F}}_{iII}[NL+i-1] = 0 \\
\qquad\qquad i = 2, 3, \ldots, NL \\
\bar{\mathbf{T}}_{NL+1} = [T_{iII}[1,1], T_{iII}[1,2], \ldots, T_{iII}[1,2NL]] \\
\bar{\mathbf{T}}_{NL+i} = [T_{iII}[NL+i+1,1] - T_{iII}[NL+i-1,1], \ldots, T_{iII}[NL+i+1,NL] - T_{iII}[NL+i-1,2NL]] \\
\qquad\qquad (i = 2, 3, \ldots, NL-1) \\
\bar{\mathbf{T}}_{2NL} = [-T_{iII}[2NL-1,1], -T_{iII}[2NL-1,2], \ldots, -T_{iII}[2NL-1,2NL]]. \qquad (i = NL)
\end{cases}
\end{aligned} \tag{39}$$

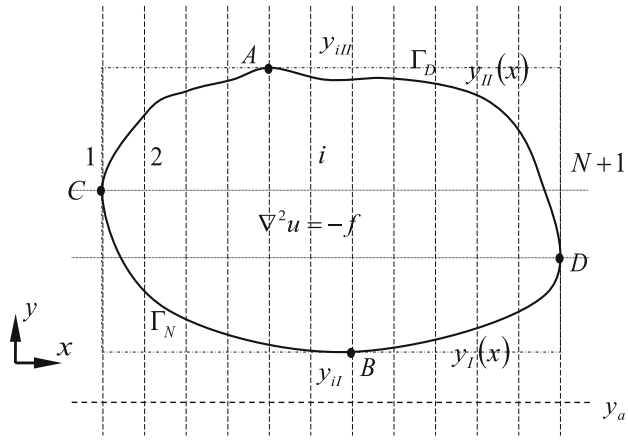


Fig. 4. PIMOL mesh for an arbitrary irregular domain

Rearranging Eq. (39) gives

$$\begin{aligned} &\tilde{\mathbf{T}}\mathbf{U}_{y_a} + \tilde{\mathbf{F}} = 0, \qquad \tilde{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}} \\ \bar{\mathbf{T}} \end{bmatrix} \\ &\mathbf{U}_{y_a} = \left[v_1\left(y_a\right), v_2\left(y_a\right), \ldots, v_{NL}\left(y_a\right), u_1\left(y_a\right), u_2\left(y_a\right), \ldots, u_{NL}\left(y_a\right)\right]^{\mathrm{T}} = -\tilde{\mathbf{T}}^{-1}\tilde{\mathbf{F}} \end{aligned} \tag{40}$$

The problem is now solved. For any point (x, y) in the domain in Fig. 3c, the semi-analytic solutions with respect to y on each line can be obtained by using Eq. (35); for any x falling out of the mesh lines, the interpolation method among many others can be used to obtain a relatively accurate solution (Fig. 4).

4. PIMOL: Arbitrary Irregular Domain

Special treatment 1: Γ_N for all BCs

It is an underdetermined problem because all Neumann BCs Γ_N cannot eliminate the rigid body displacement, namely the Dirichlet BCs at n points are needed for a n-dimensional problem at least.

Special treatment 2: Γ_D for all BCs

It is a well-posed problem.

Special treatment 3: $\Gamma_D + \Gamma_N$ for all BCs

It is also a well-posed problem. Because there are at least two overlapped joint points between different BCs, we only need to mesh at least two lines at the overlapped joint points.

General case:

Simply treating the degenerated lines on the left point C and right point D as Neumann or Dirichlet BCs as before, and introducing an auxiliary end line $y = y_a$, the solution can be obtained as in Sect. 3.2.

5. Numerical Examples

We use the following three examples, calculated by the self-programming program with the computer package Maple 17.00 [22], to explore the precision and efficiency of this new semi-analytical algorithm of PIMOL. The focus is on the numerical integration part in Eq. (24) for a given y. In the present paper, the Gaussian integral method is adopted to guarantee the convergence of a large-scale matrix $\mathbf{A}$.

Before any numerical examples are given, let us remark that, as a semi-discrete method, the discretization errors introduced in the FDMOL formulation are limited to the x-direction in terms of h, as long as the associated ODE system can be accurately solved based on the PIM. Analytical solutions are obtained along the mesh line by ignoring the error from the numerical integration.

Example 1 Torsion of a square bar with regular solving region

Let $a = b = 1,\ f(x, y) = 2$. The corresponding physical model is an elastic torsion of a square bar or a small deflection of a square membrane subject to a uniform transverse load. The PIMOL meshing

is shown in Fig. 2. The computational results are given in Table 1. Comparing with the analytical solution, it can be seen that the accuracy is satisfactory and the convergence of u is indeed within the order of h^2, as shown in Fig. 5.

Example 2 Torsion of a square bar with irregular solving region

Just like Example 1 with $a = b = 1$, $f(x, y) = 2$, the corresponding physical model is an elastic torsion of a square bar or a small deflection of a square membrane subject to a uniform transverse load. Using bi-directional and diagonal symmetry, we solve this problem on a trapezoidal region of the domain, as shown in Fig. 6. The computational results are given in Table 2. Although the present method best suits the domains with two parallel straight boundaries as boundary lines in Fig. 6, we can still see that the accuracy is satisfactory and the convergence of u is indeed within the order of h^2, as shown in Fig. 7.

Example 3 Torsion of a circular bar

Table 1. **PIMOL solution on a square domain**

NI	h	$u\big\|_o$	e (%) (error)	$\frac{e}{100h^2}$	$\frac{\partial u}{\partial y}\Big\|_A$	e (%) (error)	$\frac{e}{100h^2}$
2	0.5	0.575568	2.346739	0.0939	− 1.324137	1.988389	0.0795
3	1/3	0.583066	1.074734	0.0967	− 1.338561	0.920755	0.0829
4	0.25	0.585789	0.612669	0.0980	− 1.343780	0.534442	0.0855
5	0.2	0.587068	0.395685	0.0989	− 1.346227	0.353291	0.0883
6	1/6	0.587768	0.276975	0.0997	− 1.347565	0.254252	0.0915
7	1/7	0.588191	0.205108	0.1005	− 1.348375	0.194315	0.0952
8	0.125	0.588467	0.158347	0.1013	− 1.348902	0.155324	0.0994
9	1/9	0.588656	0.126234	0.1022	− 1.349263	0.128552	0.1041
10	0.1	0.588792	0.103237	0.1032	− 1.349522	0.109382	0.1094
11	1/11	0.588892	0.086208	0.1043	− 1.349714	0.095187	0.1152
12	1/12	0.588968	0.073248	0.1055	− 1.349860	0.084384	0.1215
13	1/13	0.589028	0.063157	0.1067	− 1.349974	0.075973	0.1284
14	1/14	0.589075	0.055147	0.0932	1.350064	0.069276	0.1171
15	1/15	0.589113	0.048683	0.0823	1.350114	0.065564	0.1108
16	1/16	0.589144	0.043392	0.0733	1.350211	0.058437	0.0988
Analytic [23]		0.5894			− 1.351		

NL = NI + 1, NI-No. of intervals, NL-No. of lines

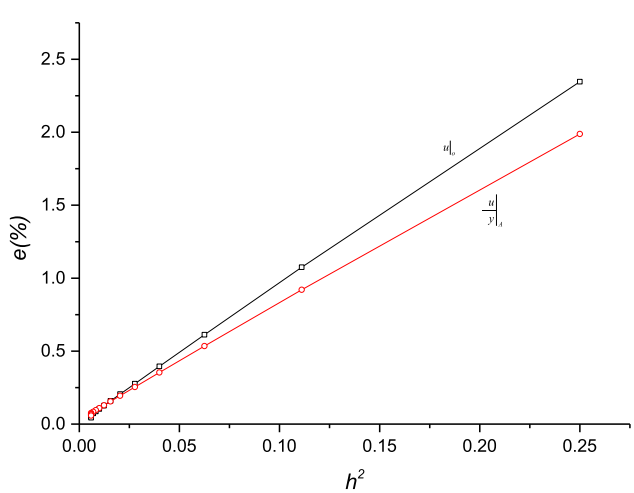


Fig. 5. The error of displacement and its derivative along h^2

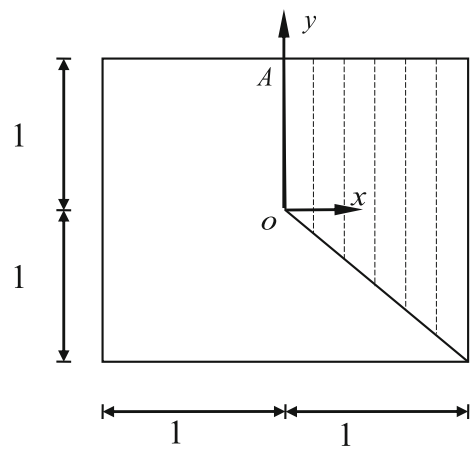


Fig. 6. PIMOL line meshes or trapezoidal region for a square bar

Table 2. **PIMOL solution on a trapezoidal region**

NI	h	$u\|_o$		$\frac{e}{100h^2}$	$\left.\frac{\partial u}{\partial y}\right\|_A$		$\frac{e}{100h^2}$
		PIMOL	[15]		PIMOL	[15]	
2	0.5	0.572797	0.5837	0.1127	− 1.321032	− 1.3421	0.0887
3	1/3	0.581246		0.1245	− 1.336716		0.0952
4	0.25	0.584574		0.1310	− 1.342624		0.0992
5	0.2	0.586228		0.1345	− 1.345462		0.1025
6	1/6	0.587141		0.1380	− 1.347010		0.1063
7	1/7	0.587733		0.1386	− 1.347971		0.1099
8	0.125	0.588083		0.1430	− 1.348573		0.1150
9	1/9	0.588372		0.1412	− 1.349025		0.1184
10	0.1	0.588515	0.5882	0.1501	− 1.349289	− 1.3487	0.1267
Analytic [23]		0.5894			− 1.351		

NL = NI + 1, NI-No. of intervals, NL-No. of lines

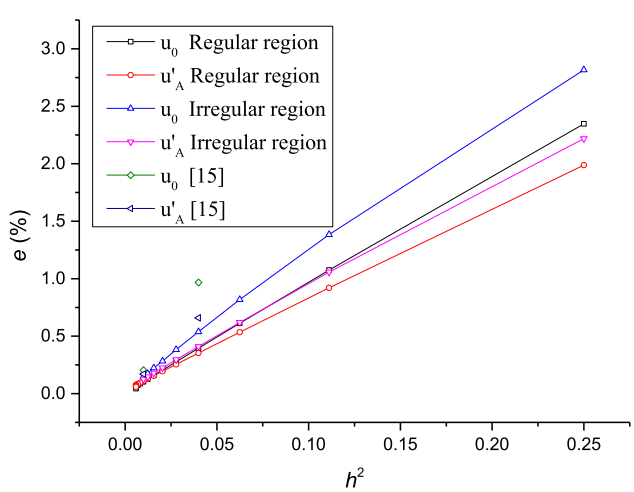


Fig. 7. The error of displacement and its derivative along h^2

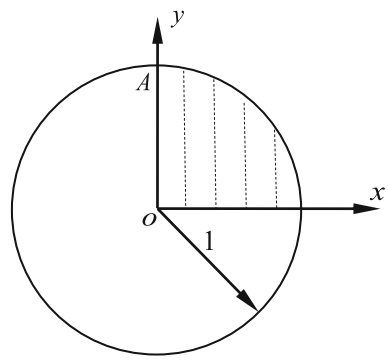


Fig. 8. PIMOL line meshes for a circular bar

Table 3. **PIMOL solution on a quarter circular region**

NI	h	$u\|_o$		e (%)	$\frac{\partial u}{\partial y}\Big\|_A$		e (%)
	1/NI	PIMOL	[15]	(error)	PIMOL	[15]	(error)
2	1/2	0.529233	0.50368	5.847	− 1.031530	− 1.00418	3.153
3	1/3	0.519022		3.804	− 1.023028		2.303
4	1/4	0.512815		2.563	− 1.013320		1.332
5	1/5	0.509106		1.821	− 1.009453		0.945
6	1/6	0.506831		1.366	− 1.007037		0.704
7	1/7	0.505370		1.074	− 1.005543		0.554
8	1/8	0.504377		0.875	− 1.004507		0.451
9	1/9	0.503662		0.732	− 1.003758		0.376
10	1/10	0.503122	0.50127	0.624	− 1.003192	− 1.00134	0.319
11	1/11	0.502700		0.540	− 1.002746		0.275
12	1/12	0.502364		0.473	− 1.002408		0.241
13	1/13	0.502091		0.418	− 1.002128		0.213
14	1/14	0.501868		0.374	− 1.001899		0.190
15	1/15	0.501682		0.336	− 1.001708		0.171
16	1/16	0.501524		0.305	− 1.001547		0.155
17	1/17	0.501390		0.278	− 1.001399		0.140
18	1/18	0.501275		0.255	− 1.001291		0.129
19	1/19	0.501174		0.235	− 1.001189		0.119
20	1/20	0.501086	0.50044	0.217	− 1.001099	− 1.00046	0.110
Analytic [23]		0.5			− 1.0		

NL = NI + 1, NI-No. of intervals, NL-No. of lines

Figure 8 shows the cross section of a circular bar and the line mesh for PIMOL analysis. With the knowledge that the exact solution is a quadratic polynomial

$$u = \left(1 - x^2 - y^2\right)/2 \tag{41}$$

this example has been selected to demonstrate the applicability of the proposed method to irregular domains including curved boundaries (Table 3; Figs. 9, 10).

6. Conclusions

A new semi-analytical method solving BVPs of elliptic type is presented. Three examples of BVPs of elliptic type are given. The numerical results show that PIMOL is a powerful method. It is of great value that PIMOL can reduce a semi-discrete BVP to a linear algebraic matrix equation problem. Based on the numerical experimentation discussed above, the following conclusions can be drawn:

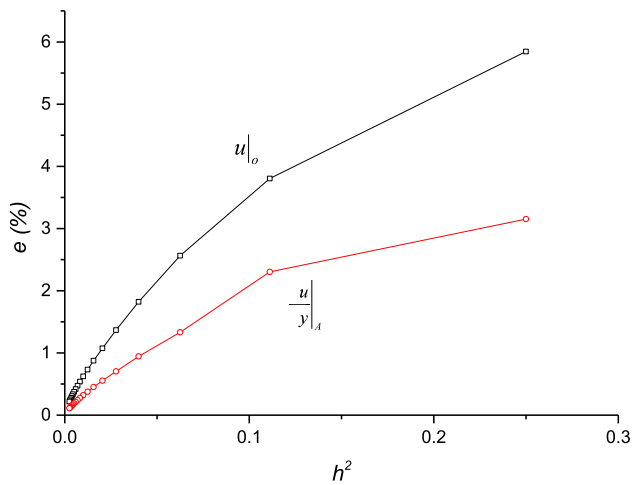


Fig. 9. The error of displacement and its derivative along h^2

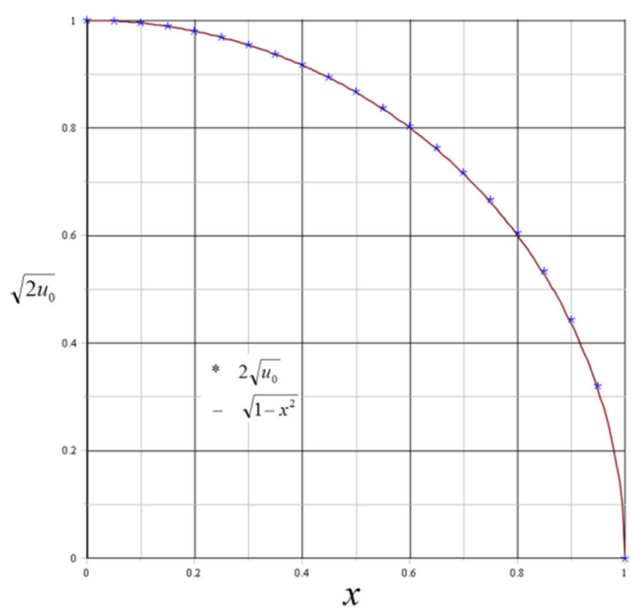


Fig. 10. A comparison of the PIMOL solution (NI = 20) with the analytical solution

(1) **New semi-analytical method** The present method PIMOL is a newly developed semi-analytical method for elliptic BVPs. In this method, the PDEs defined on arbitrary domains are semi-discretized by the MOL into a system of ODEs defined on discrete mesh lines, and then, the analytical result is expressed in algebraic matrix equations using the precise integration method. The PIMOL completely changes the PDEs of elliptic type into solving a linear algebraic matrix equation.

(2) **Generality** PIMOL is not restricted to the Poisson's equation problems. It can be easily extended to plane problems, plate-and-shell problems, 3D problems, and so on. It can also be extended to other subjects such as the pipes conveying fluid, fluid–structure interaction, multi-body dynamics, and so on. It can be also easily extended to the parametric finite difference method of lines and finite element method of lines.

(3) **Accuracy** Theoretically, PIMOL is a semi-analytical method. Such high precision of the results is guaranteed by semi-discrete approximation of MOL. However, the analytical expression cannot be

computed directly, and numerical integration with PIM is inevitable. Fortunately, several numerical algorithms can guarantee desirable accuracy of the solutions.

(4) **Reliability** The comparisons have shown that the results agree very well with the exact solutions.

(5) **Efficiency** The present work has demonstrated that PIMOL has high precision and computational efficiency in solving the PDEs of elliptic type

(6) With PIMOL, irregular domains are no longer needed to be mapped to regular ones when treating arbitrary irregular domain problems.

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