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Article

Sedimentation motion of sand particles in moving water (I): The resistance on a small sphere moving in non-uniform flow☆,☆☆

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A R T I C L E I N F O

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E D I T O R - I N - C H E I F ' S R E C O M M E N D A T I O N

The Maxey-Riley equation, which describes the force and motion of a single small-Reynolds-number spherical particle in an unsteady and non-uniform flow field, forms the basis for the study of particulate two-phase flows. The equation was published by Martin R. Maxey and James J. Riley in Physics of Fluids (26(4), 883-889) in 1983 and has had an important impact on the following research (as of August 29, 2022, the paper has been cited by 2250 times, including 129 citations for year 2021).

However, as early as 1956-1957, when Prof. Shu-tang Tsai worked on the sediment settlement in rivers under the guidance of Prof. Pei-Yuan Chou, he conducted in-depth research on the basic hydrodynamic problem related to the force of spherical particles in the flow field, and published a series of results in the Acta Physica Sinica in Chinese, including the non-uniform sedimentation motion of particles in hydrostatic water (1956, 12(5), 409-418), the force on particles in arbitrary flow field (1957, 13(5), 388-398, henceforth referred as 1957a), and sediment settlement in laminar flows (1957, 13(5), 399-408). Among them, the title of the 1957a paper is "Sedimentation motion of sand particles in moving water (I) The resistance on a small sphere moving in non-uniform flow". In this paper, the equation governing the force and motion of a single small spherical particle under general flow conditions was derived, and is essentially identical to what is now known as the Maxey-Riley equation.

Since Mr. Tsai's paper was published in Chinese, it has long been unknown to the international academic community. In this issue, we publish the English version of the 1957a paper, which was translated by Prof. Haitao Xu from Tsinghua University, to memorize Prof. Shu-tang Tsai and let more people aware of his pioneering work in the field of particulate two-phase flows.

A B S T R A C T

In hydraulics, when we deal with the problem of sand particles moving relative to the surrounding water, Stokes' formula of resistance has usually been used to render the velocity of sedimentation of the particles. But such an approach has not been proved rigorously, and its accuracy must be carefully considered. In this paper, we discuss the problem of a sphere moving in a non-uniform flow field, on the basis of the fundamental theory of hydrodynamics. We introduce two assumptions: i) the diameter of the sphere is much smaller than the linear dimension of the flow field, and ii) the velocity of the sphere relative to the surrounding water is very small. Using these two assumptions, we solve the linearized Navier-Stokes equations and equations of continuity by the method of Laplace transform, and finally we obtain a formula for the resistance acting on a sphere moving in a non-uniform flow field.

☆ Original article: Shu-Tang Tsai, Sedimentation motion of sand particles in moving water (II)——Sedimentation of sand particles in laminar flow, Acta Phys. Sin. 13 (1957) 399-408. doi: 10.7498/aps.13.399. The manuscript was received on March 27, 1957. This translation, by Yuchen Luo and Haitao Xu from Tsinghua University, Beijing, China, was reprinted here on the occasion of the 65th year of the publication of the original article, in memory of the contribution of Prof. Shu-Tang Tsai, who passed away in August 2008, to the development of particle-dynamics in fluid flows. The original article is the first of a two-paper series. The second article of the series is on the application of the theoretical result obtained in the first to the case of sand particles sedimenting in laminar water flows and is not translated here.

☆☆ The original article (Tsai, 1957) contains a title and an abstract in English, which are reprinted here.

[1] The author Tsai Shu-Tang passed away.

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1. Introduction

In the study of river-bed dynamics [1], when dealing with the relative motion between sand particles and the surrounding water, Stokes' drag formula is usually used to determine the relative velocities of the sand particles in water. The validity of this approach, however, has not been rigorously established and thus must be carefully discussed. In two previous articles [2], we discussed the sedimentation motion of sand particles in quiescent water, and obtained the correction to Stokes' formula for calculating the resistant forces on sand particles under this situation. To further study the sedimentation of sand particles in moving water, in the present paper we start from the fundamental theory of hydrodynamics and analyze the resistance forces on a sphere moving in a general flow field. We obtain a formula for the total resistance force. With the help of that, we will be able to study the relative velocities between sand particles and the surrounding water, and to understand the sedimentation process of sand particles in rivers.

2. Derivation of the resistance formula

2.1. Decomposition of the velocity and the drag force

Consider a flow field consisting of a viscous fluid in space, with proper initial and boundary conditions on the relevant hydrodynamic variables such as velocity and pressure. Suppose we now put a sphere with radius a into the flow field, while keeping the initial and the boundary conditions of the flow in regions far away from the sphere unchanged, then a new flow state is established, which we would like to analyze. We take a coordinate system moving with the sphere, with the origin on the center of the sphere. We use Cartesian tensors to describe the motion. We denote the velocity of the sphere as v_i and the rotation tensor of the sphere as ω_{ij}. Before the sphere is put into the flow field, the momentum equation and the continuity equation of the fluid are, respectively,

$$\frac{\partial u_i^0}{\partial t} + u_\alpha^0 \frac{\partial u_i^0}{\partial x_\alpha} + \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p^0}{\partial x_i} + \nu\nabla^2 u_i^0 - g_i \tag{1}$$

and

$$\frac{\partial u_i^0}{\partial x_i} = 0, \tag{2}$$

where u_i^0 is the velocity of the fluid relative to the moving coordinates when the sphere is not in the flow field, p^0 is the fluid pressure, ρ is the fluid density, ν is the kinematic viscosity of fluid, and g_i is gravitational acceleration. The whole flow field changes after the sphere is introduced into the flow. Let u_i denote the velocity of the fluid relative to the moving coordinates after introducing the sphere to the flow field and p denote the corresponding pressure, then the momentum equation and the continuity equation for that flow field are, respectively,

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} + \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i - g_i \tag{3}$$

and

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{4}$$

If we define $W_i = u_i - u_i^0$ and $\varpi = p - p^0$, the equations for W_i and ϖ can be obtained by subtracting Eq. (3) from Eqs. (1) and (4) from Eq. (2), respectively. The results are

$$\frac{\partial W_i}{\partial t} + u_\alpha \frac{\partial W_i}{\partial x_\alpha} + W_\alpha \frac{\partial u_i^0}{\partial x_\alpha} = -\frac{1}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu\nabla^2 W_i \tag{5}$$

and

$$\frac{\partial W_i}{\partial x_i} = 0. \tag{6}$$

Assuming that the velocity of the fluid relative to the sphere center is very small, we can neglect the nonlinear terms in Eq. (5), which then becomes

$$\frac{\partial W_i}{\partial t} = -\frac{1}{\rho}\frac{\partial \varpi}{\partial x_i} + \nu\nabla^2 W_i. \tag{7}$$

Next we discuss the initial condition and the boundary conditions that W_i needs to satisfy. For the initial condition, we assume that right after introducing the sphere to the flow field, the velocity of the fluid outside the sphere remains the same as that in the original flow field without the sphere, i.e., at any position $\mathbf{x}$ with $|\mathbf{x}| > a$,

$$W_i = 0 \quad \text{at} \quad t = 0. \tag{8}$$

For the boundary conditions, we first note that far from the sphere, the velocity disturbance caused by the sphere should vanish, which means that

$$W_i = 0 \quad \text{at} \quad |\mathbf{x}| = \infty. \tag{9}$$

We also note that on the surface of the sphere, because the fluid is viscous, the fluid velocity must be the same as the rotational velocity of the surface, which gives

$$u_i = \omega_{ij} x_j \quad \text{at} \quad |\mathbf{x}| = a, \tag{10}$$

which could be written as

$$W_i = \omega_{ij} x_j - u_i^0 \quad \text{at} \quad |\mathbf{x}| = a. \tag{11}$$

We now expand the undisturbed velocity field u_i^0 at the origin of the moving coordinate system as

$$u_i^0 = \alpha_i + \alpha_{ij} x_j + \alpha_{ijk} x_j x_k + \mathcal{O}\left(\frac{a^3}{L^3}\right),$$

in which α_i, α_{ij}, α_{ijk} are all functions of time t, L is the linear length scale of the flow field, and $\mathcal{O}\left(a^3/L^3\right)$ represents the sum of all terms with orders above or equal to $\left(a^3/L^3\right)$. Because the radius of the sphere a is much smaller than the length scale of the flow field, we neglect all terms of order above $\left(a^2/L^2\right)$ and simplify the expansion to

$$u_i^0 = \alpha_i + \alpha_{ij} x_j + \alpha_{ijk} x_j x_k. \tag{12}$$

Due to the continuity Eq. (2), α_{ij} and α_{ijk} must satisfy

$$\alpha_{jj} = 0 \quad \text{and} \quad \alpha_{jjk} = 0. \tag{13}$$

In addition, u_i^0 is the solution of the momentum Eq. (1), thus α_i, α_{ij}, and α_{ijk} must satisfy other constraints, which we shall explore later. Substituting Eq. (12) into Eq. (11), we may write the final form of the boundary condition on the surface of the sphere as

$$W_i = -\alpha_i + (\omega_{ij} - \alpha_{ij}) x_j - \alpha_{ijk} x_j x_k. \tag{14}$$

In summary, the problem is transformed into solving Eqs. (6) and (7) under the initial condition Eq. (8) and the boundary conditions Eqs. (9) and (14).

In order to move forward, we take advantage of the linearity of the equations and decompose the velocity field W_i into three parts

$$W_i = u_i' + u_i'' + u_i''', \tag{15}$$

with each of the three parts separately satisfying the following equations and initial and boundary conditions:

1. Equations and conditions for u_i'
   The governing equations for u_i' are

$$\frac{\partial u_i'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 u_i', \tag{16}$$

and

$$\frac{\partial u_i'}{\partial x_i} = 0, \tag{17}$$

where p' is the pressure corresponding to u_i'.
The initial condition for u_i' is

$$u_i' = 0 \quad \text{when} \quad t = 0. \tag{18}$$

The boundary conditions for u_i' are

$$\left.\begin{array}{lll} u_i' = 0 & \text{at} & |\mathbf{x}| = \infty \\ u_i' = -\alpha_i & \text{at} & |\mathbf{x}| = a \end{array}\right\} \tag{19}$$

2. Equations and conditions for u_i''
The equations for u_i'' are

$$\frac{\partial u_i''}{\partial t} = -\frac{1}{\rho}\frac{\partial p''}{\partial x_i} + \nu\nabla^2 u_i'', \tag{20}$$

and

$$\frac{\partial u_i''}{\partial x_i} = 0, \tag{21}$$

where p'' is the pressure corresponding to u_i''.
The initial condition for u_i'' is

$$u_i'' = 0 \quad \text{when} \quad t = 0. \tag{22}$$

The boundary conditions for u_i'' are

$$\left.\begin{array}{lll} u_i'' = 0 & \text{at} & |\mathbf{x}| = \infty, \\ u_i'' = (\omega_{ij} - \alpha_{ij})x_j & \text{at} & |\mathbf{x}| = a. \end{array}\right\} \tag{23}$$

3. Equations and conditions for u_i'''
The equations for u_i''' are

$$\frac{\partial u_i'''}{\partial t} = -\frac{1}{\rho}\frac{\partial p'''}{\partial x_i} + \nu\nabla^2 u_i''', \tag{24}$$

and

$$\frac{\partial u_i'''}{\partial x_i} = 0, \tag{25}$$

where p''' is the pressure corresponding to u_i'''.
The initial condition for u_i''' is

$$u_i''' = 0 \quad \text{when} \quad t = 0. \tag{26}$$

The boundary conditions for u_i''' are

$$\left.\begin{array}{lll} u_i''' = 0 & \text{at} & |\mathbf{x}| = \infty, \\ u_i''' = -\alpha_{ijk}x_j x_k & \text{at} & |\mathbf{x}| = a. \end{array}\right\} \tag{27}$$

Let f_i denote the total force exerted by the fluid on the sphere, which can be written in terms of the fluid stress as

$$f_i = \oiint_S \tau_{i\alpha}\frac{x_\alpha}{a}\mathrm{d}S, \tag{28}$$

in which $\oiint_S dS$ represents integration over the surface of the sphere and $\tau_{i\alpha}$ is the fluid stress on the surface of the sphere, which has the form

$$\tau_{i\alpha} = \mu\left(\frac{\partial u_i}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_i}\right) - p\delta_{i\alpha} \tag{29}$$

with μ the dynamic viscosity coefficient of the fluid.

The total force acting on the sphere f_i can be decomposed into two parts: $f_i = f_i^0 + \varphi_i$, in which f_i^0 corresponds to the velocity field u_i^0 and φ_i corresponds to the velocity field W_i. Thus the expressions of f_i^0 and φ_i are

$$f_i^0 = /_S \tau_{i\alpha}^0 \frac{x_\alpha}{a}\mathrm{d}S, \tag{30}$$

where

$$\tau_{i\alpha}^0 = \mu\left(\frac{\partial u_i^0}{\partial x_\alpha} + \frac{\partial u_\alpha^0}{\partial x_i}\right) - p^0\delta_{i\alpha}, \tag{31}$$

and

$$\varphi_i = /_S \varpi_{i\alpha}\frac{x_\alpha}{a}\mathrm{d}S, \tag{32}$$

where

$$\varpi_{i\alpha} = \mu\left(\frac{\partial W_i}{\partial x_\alpha} + \frac{\partial W_\alpha}{\partial x_i}\right) - \varpi\,\delta_{i\alpha}. \tag{33}$$

Similarly, we can further divide φ_i into three contributions: $\varphi_i = f_i' + f_i'' + f_i'''$, in which f_i' comes from u_i' and can be calculated as

$$f_i^{'} = /_S \tau_{i\alpha}^{'}\frac{x_\alpha}{a}\mathrm{d}S, \tag{34}$$

where

$$\tau_{i\alpha}' = \mu\left(\frac{\partial u_i'}{\partial x_\alpha} + \frac{\partial u_\alpha'}{\partial x_i}\right) - p'\delta_{i\alpha}, \tag{35}$$

f_i'' corresponds to u_i'':

$$f_i^{''} = /_S \tau_{i\alpha}^{''}\frac{x_\alpha}{a}\mathrm{d}S, \tag{36}$$

where

$$\tau_{i\alpha}'' = \mu\left(\frac{\partial u_i''}{\partial x_\alpha} + \frac{\partial u_\alpha''}{\partial x_i}\right) - p''\delta_{i\alpha}, \tag{37}$$

and f_i''' corresponds to u_i''':

$$f_i^{'''} = /_S \tau_{i\alpha}^{'''}\frac{x_\alpha}{a}\mathrm{d}S, \tag{38}$$

where

$$\tau_{i\alpha}''' = \mu\left(\frac{\partial u_i'''}{\partial x_\alpha} + \frac{\partial u_\alpha'''}{\partial x_i}\right) - p'''\delta_{i\alpha}. \tag{39}$$

2.2. Calculation of f_i^0, f_i' and f_i''

To obtain an explicit expression for the contribution f_i^0 to the total resistance, we use Gauss' divergence theorem to transform Eq. (30) into

$$f_i^0 = \iiint_V \frac{\partial \tau_{i\alpha}^0}{\partial x_\alpha}\mathrm{d}V = \iiint_V\left(\mu\nabla^2 u_i^0 - \frac{\partial p^0}{\partial x_i}\right)\mathrm{d}V, \tag{40}$$

in which $\iiint_V \mathrm{d}V$ represents integration over the volume occupied by the sphere. With the help of Eq. (1), Eq. (40) becomes

$$f_i^0 = \rho\iiint_V\left(\frac{\partial u_i^0}{\partial t} + u_\alpha^0\frac{\partial u_i^0}{\partial x_\alpha} + \frac{d\nu_i}{dt} + g_i\right)\mathrm{d}V. \tag{41}$$

Because the radius of the sphere is very small, the integral can be evaluated by replacing the function in the integrand with its value at the origin of coordinate system, or the center of the sphere, which introduces an error of the order higher than $\left(a^3/L^3\right)$ and is thus acceptable. The result is

$$\begin{aligned} f_i^0 &= \frac{4}{3}\pi a^3\rho\left[\left(\frac{\partial u_i^0}{\partial t} + u_\alpha^0\frac{\partial u_i^0}{\partial x_\alpha}\right)_{r=0} + \frac{\mathrm{d}\nu_i}{\mathrm{d}t} + g_i\right] \\ &= \frac{4}{3}\pi a^3\rho\left(\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} + \frac{\mathrm{d}\nu_i}{\mathrm{d}t} + g_i\right), \end{aligned} \tag{42}$$

in which $()_{r=0}$ means the value evaluated at the origin or the center of the sphere.

To evaluate the contribution f_i' to the drag force, we note that it follows directly from the work by Boussinesq [3,4] as

$$f_i' = 6\pi\mu a\left(\alpha_i + \frac{a}{\sqrt{\pi\nu}}\int_0^t \frac{\frac{\mathrm{d}\alpha_i}{\mathrm{d}\tau}}{\sqrt{t-\tau}}\mathrm{d}\tau\right) + \frac{2}{3}\pi a^3\rho\frac{\mathrm{d}\alpha_i}{\mathrm{d}t}. \tag{43}$$

Table 1
Laplace transform pairs.

Original function	Laplace transformation
t	s
u_i'''	U_i
$-\alpha_{ijk}$	β_{ijk}
p'''	P
τ_{ij}'''	T_{ij}
f_i'''	F_i

We now turn to f_i''. We note that Eqs. (20) and (21), together with the initial condition Eq. (22) and boundary conditions Eq. (23), indicate that u_i'' must be an odd function of x_j, and p'' is an even function of x_j. Therefore, the integrand in Eq. (36) is an odd function of x_j. Hence the integration over the entire surface of the sphere is identically zero, which means that f_i'', the contribution corresponding to velocity u_i'' and pressure p'', is 0.

2.3. Calculation of f_i'''

To evaluate the force f_i''', instead of solving Eqs. (24) and (25) directly, we resort to the method of Laplace transform. For a given function $\varphi(t)$, its Laplace transform $\Phi(s)$ is given by

$$\Phi(s) = \int_0^\infty \mathrm{e}^{-st}\varphi(t)\mathrm{d}t.$$

In Table 1, we list the symbols representing the pairs of original functions and their Laplace transforms that we will use later.

Taking the Laplace transform of Eqs. (24) and (25), we obtain

$$sU_i = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\nabla^2 U_i \tag{44}$$

and

$$\frac{\partial U_i}{\partial x_i} = 0. \tag{45}$$

The boundary conditions for the Laplace-transformed function are

$$\left.\begin{array}{lll} U_i = 0 & \text{at} & |\mathbf{x}| = \infty \\ U_i = \beta_{ijk}x_jx_k & \text{at} & |\mathbf{x}| = a. \end{array}\right\} \tag{46}$$

The continuity Eq. (45) requires that β_{ijk} must satisfy

$$\beta_{ijj} = 0. \tag{47}$$

The initial condition does not appear explicitly here since the original initial condition Eq. (26) has been used in the Laplace transformation and thus is already included in Eqs. (44) and (45).

Before solving Eqs. (44) and (45), let's consider a third-order tensor W_{imn} that satisfies

$$sW_{imn} = -\frac{1}{\rho}\frac{\partial P_{mn}}{\partial x_i} + \nu\nabla^2 W_{imn} \tag{48}$$

and

$$\frac{\partial}{\partial x_i}W_{imm} = 0, \tag{49}$$

with the boundary conditions

$$\left.\begin{array}{lll} W_{imn} = 0 & \text{at} & |\mathbf{x}| = \infty \\ W_{imn} = \beta_{imn} & \text{at} & |\mathbf{x}| = a. \end{array}\right\} \tag{50}$$

To solve Eqs. (48) and (49), we let R denote the distance of a point in space to the origin of the coordinate system, i.e., $R = |\mathbf{x}|$, and introduce a new variable $\eta = R^2$. We seek solutions of the following forms:

$$W_{imn} = f(\eta)\beta_{lmn}x_ix_l + g(\eta)\beta_{imn} \tag{51}$$

and

$$\frac{1}{\rho}P_{mn} = A\eta^{-\frac{3}{2}}\beta_{lmn}x_l, \tag{52}$$

in which A is a constant to be determined, $f(\eta)$ and $g(\eta)$ are functions of η only. For simplicity in writing, later we will use f and g to denote $f(\eta)$ and $g(\eta)$, and use a superscript "$'$" on the upper-right corner to denote differentiation with respect to η once. Substituting Eq. (51) into Eq. (49), we obtain

$$\frac{\partial W_{imn}}{\partial x_i} = (2\eta f' + 4f + 2g')\beta_{lmn}x_l = 0,$$

which leads to

$$2\eta f' + 4f + 2g' = 0. \tag{53}$$

Differentiating Eqs. (51) and (50) gives

$$\nabla^2 W_{imn} = (4\eta f'' + 14f')\beta_{jmn}x_ix_j + (2f + 2\eta g'' + 6g')\beta_{imn}$$

and

$$\frac{1}{\rho}\frac{\partial P_{mn}}{\partial x_i} = -3A\eta^{-\frac{5}{2}}\beta_{jmn}x_ix_j + A\eta^{-\frac{3}{2}}\beta_{imn},$$

which, upon substituting into Eq. (48), lead to the following two equations:

$$sf = 3A\eta^{-\frac{5}{2}} + \nu(4\eta f'' + 14f') \tag{54}$$

and

$$sg = -A\eta^{-\frac{3}{2}} + \nu\left(2f + 2\eta g'' + 6g'\right). \tag{55}$$

Note that among Eqs. (53), (54) and (55), only two are independent and the third can be obtained from the other two. Here we choose to solve Eqs. (53) and (54). For easy writing, we use m to denote $\sqrt{\frac{s}{\nu}}$. Upon changing the independent variable in Eqs. (53) and (54) from η to R, we obtain

$$\left.\begin{array}{l} \dfrac{\mathrm{d}^2 f}{\mathrm{d}R^2} + \dfrac{6}{R}\dfrac{\mathrm{d}f}{\mathrm{d}R} - m^2 f + \dfrac{3A}{\nu}\dfrac{1}{R^5} = 0 \\ \dfrac{1}{R}\dfrac{\mathrm{d}g}{\mathrm{d}R} = -4f - R\dfrac{\mathrm{d}f}{\mathrm{d}R}, \end{array}\right\} \tag{56}$$

which should be solved with proper boundary conditions. Far away from the sphere, W_{imn} vanishes, hence

$$f(\infty) = g(\infty) = 0. \tag{57}$$

On the surface of the sphere, $W_{imn} = \beta_{imn}$, which gives

$$f(a^2) = 0 \quad \text{and} \quad g(a^2) = 1. \tag{58}$$

Solving Eq. (56) with the boundary conditions Eqs. (57) and (58), we obtain

$$\left.\begin{array}{l} f = -\dfrac{9a}{2m^2R^5}\mathrm{e}^{-m(R-a)}\left(1 + mR + \dfrac{1}{3}m^2R^2\right) \\ \qquad + \dfrac{9a}{2m^2R^5}\left(1 + ma + \dfrac{1}{3}m^2a^2\right), \\ g = \dfrac{3a}{2m^2R^3}\mathrm{e}^{-m(R-a)}(1 + mR + m^2R^2) \\ \qquad - \dfrac{3a}{2m^2R^3}\left(1 + ma + \dfrac{1}{3}m^2a^2\right), \\ \dfrac{A}{\nu} = \dfrac{3}{2}a\left(1 + ma + \dfrac{1}{3}m^2a^2\right). \end{array}\right\} \tag{59}$$

Now we are ready to solve Eqs. (44) and (45). We first define the following pairs of functions

$$\left.\begin{aligned}(U_i)_I &= \frac{\partial^2}{\partial x_m \partial x_n}W_{imn} = 4f''\beta_{lmn}x_i x_l x_m x_n \\ &\quad +4f'\beta_{lim}x_l x_m + 2f'\beta_{lmm}x_i x_l \\ &\quad +4g''\beta_{imn}x_m x_n + 2g'\beta_{imm}, \\ (P)_I &= \frac{\partial^2}{\partial x_m \partial x_n}P_{mn} = \\ &\quad 15\rho A\eta^{-\frac{7}{2}}\beta_{lmn}x_m x_n - 3\rho A\eta^{-\frac{5}{2}}\beta_{lmm}x_l,\end{aligned}\right\} \tag{60}$$

$$\left.\begin{aligned}(U_i)_{II} &= \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}\left(\frac{\partial^2}{\partial x_m \partial x_n}W_{imn}\right) = \\ &\quad \left(4m^2 f'' - \frac{105A}{\nu}\eta^{-\frac{9}{2}}\right)\beta_{lmn}x_i x_l x_m x_n \\ &\quad +\left(4m^2 f' + \frac{30A}{\nu}\eta^{-\frac{7}{2}}\right)\beta_{lim}x_l x_m \\ &\quad +\left(2m^2 f' + \frac{15A}{\nu}\eta^{-\frac{7}{2}}\right)\beta_{lmm}x_l x_i \\ &\quad +\left(4m^2 g'' + \frac{15A}{\nu}\eta^{-\frac{7}{2}}\right)\beta_{ilm}x_l x_m \\ &\quad +\left(2m^2 g' - \frac{3A}{\nu}\eta^{-\frac{5}{2}}\right)\beta_{imm}, \\ (P)_{II} &= \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}\left(\frac{\partial^2 P_{mn}}{\partial x_m \partial x_n}\right) = 0,\end{aligned}\right\} \tag{61}$$

$$\left.\begin{aligned}(U_i)_{III} &= W_{imm} = f\beta_{lmm}x_i x_l + g\beta_{imm}, \\ (P)_{III} &= P_{mm} = \rho A\eta^{-\frac{3}{2}}x_l\beta_{lmm},\end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned}(U_i)_{IV} &= \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}W_{imn} = \left(m^2 f - \frac{3A}{\nu}\eta^{-\frac{5}{2}}\right)\beta_{lmm}x_i x_l \\ &\quad +\left(m^2\ g + \frac{A}{\nu}\eta^{-\frac{3}{2}}\right)\beta_{imm}, \\ (P)_{IV} &= \frac{\partial^2}{\partial x_\alpha \partial x_\alpha}P_{mn} = 0,\end{aligned}\right\} \tag{63}$$

and

$$\left.\begin{aligned}(U_i)_V &= h\beta_{ilm}x_l x_m - h\beta_{lim}x_l x_m + k\beta_{imm}, \\ (P)_V &= 0,\end{aligned}\right\} \tag{64}$$

in which

$$\left.\begin{aligned}h &= \frac{1}{R^5}\mathrm{e}^{-m(R-a)}\left(1 + mR + \frac{1}{3}m^2R^2\right), \\ k &= -\left(\frac{1}{3R^3} + \frac{m}{3R^2}\right)\mathrm{e}^{-m(R-a)}.\end{aligned}\right\} \tag{65}$$

Substituting the pairs of U_i and P given by Eqs. (60)-(64) into Eqs. (44) and (45), we can show that every pair of U_i and P satisfy Eqs. (44) and (45) simultaneously. Therefore, to find the solution of Eqs. (44) and (45) with the boundary conditions Eq. (46), we let

$$\left.\begin{aligned}U_i &= C_1(U_i)_I + C_2(U_i)_{II} + C_3(U_i)_{III} + C_4(U_i)_{IV} + C_5(U_i)_V, \\ P &= C_1(P)_I + C_3(P)_{III},\end{aligned}\right\} \tag{66}$$

in which C_1, C_2, C_3, C_4 and C_5 are constants to be determined. Substituting Eq. (66) into the boundary conditions Eq. (46), after some lengthy but straightforward calculation we obtain

$$\left.\begin{aligned}C_1 &= \frac{m^2a^7}{18}\frac{\nu}{A}\frac{1 + ma + \frac{1}{10}m^2a^2}{1 + ma + \frac{1}{3}m^2a^2} + \frac{7}{18}\frac{a^4}{1 + ma + \frac{1}{3}m^2a^2}, \\ C_2 &= -\frac{\nu a^7}{18A}\frac{1 + ma + \frac{1}{10}m^2a^2}{1 + ma + \frac{1}{3}m^2a^2}, \\ C_3 &= \frac{a^2}{3}\frac{1 + ma + \frac{1}{5}m^2a^2}{1 + ma + \frac{1}{3}m^2a^2}, \\ C_4 &= -\frac{1}{18}\frac{a^4}{1 + ma + \frac{1}{3}m^2a^2}, \\ C_5 &= -\frac{1}{2}\frac{a^5}{1 + ma + \frac{1}{3}m^2a^2}.\end{aligned}\right\} \tag{67}$$

Putting these together, U_i and P are given by

$$\left.\begin{aligned}U_i &= (2D_2 f'' - 35D_1R^{-9})\beta_{lmn}x_i x_l x_m x_n \\ &\quad +(2D_2 f' + 10D_1R^{-7} - D_3\ h)\beta_{lim}x_l x_m \\ &\quad +(2D_2 g'' + 5D_1R^{-7} + D_3\ h)\beta_{ilm}x_l x_m \\ &\quad +(D_2 f' + 5D_1R^{-7} + D_4 f \\ &\quad -3D_5R^{-5})\beta_{lmm}x_i x_l \\ &\quad +(D_2 g' - D_1R^{-5} + D_4\ g \\ &\quad +D_5R^{-3} + D_3 k)\beta_{imm}, \\ P &= \rho(15C_1AR^{-7}\beta_{lmn}x_l x_m x_n - 3C_1AR^{-5}\beta_{lmm}x_l \\ &\quad + C_3AR^{-3}\beta_{lmm}x_l),\end{aligned}\right\} \tag{68}$$

in which

$$\left.\begin{aligned}D_1 &= -\frac{a^7}{6}\frac{1 + ma + \frac{1}{10}m^2a^2}{1 + ma + \frac{1}{3}m^2a^2}, \\ D_2 &= \frac{7}{9}\frac{a^4}{1 + ma + \frac{1}{3}m^2a^2}, \\ D_3 &= -\frac{1}{2}\frac{a^5}{1 + ma + \frac{1}{3}m^2a^2}, \\ D_4 &= \frac{a^2}{3}\frac{1 + ma + \frac{1}{30}m^2a^2}{1 + ma + \frac{1}{3}m^2a^2}, \\ D_5 &= -\frac{1}{12}a^5.\end{aligned}\right\} \tag{69}$$

The Laplace-transformed function F_i corresponding to f_i''' is

$$F_i = \frac{1}{a} \oiint_S T_{ij} x_j \mathrm{d}S, \tag{70}$$

in which

$$T_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - P\delta_{ij}. \tag{71}$$

Therefore, substituting Eq. (68) into Eqs. (70) and (71), after some heavy but manageable manipulation we finally obtain

$$\begin{aligned} F_i = & -2\pi\mu a^3 \beta_{imm} \Big(1 + ma + \frac{1}{15} m^2 a^2 \\ & - \frac{4}{15} \frac{m^2 a^2}{1 + ma + \frac{1}{3} m^2 a^2} \Big). \end{aligned} \tag{72}$$

If terms of order above a^3 are neglected, it simplifies to

$$F_i = -2\pi\mu a^3 \beta_{imm}, \tag{73}$$

which, by the inverse Laplace transform, gives the contribution f_i''' to the total force simply as

$$f_i''' = 2\pi\mu a^3 \alpha_{imm} = \pi\mu a^3 \left(\nabla^2 u_i^0 \right)_{r=0}. \tag{74}$$

2.4. Total drag on a sphere moving in a flow field

Summing up the contributions f_i^0, f_i' and f_i''' given by Eqs. (42), (43) and (74), we obtain the total flow resistance f_i on the sphere as

$$\begin{aligned} f_i = & f_i^0 + f_i' + f_i''' = \frac{4}{3}\pi a^3 \rho \left(\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} + \frac{\mathrm{d}v_i}{\mathrm{d}t} + g_i \right) \\ & + 6\pi\mu a \left(\alpha_i + \frac{a}{\sqrt{\pi\nu}} \int_0^t \frac{\frac{\mathrm{d}\alpha_i}{\mathrm{d}\tau}}{\sqrt{t-\tau}} \mathrm{d}\tau \right) \\ & + \frac{2}{3}\pi a^3 \rho \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} + 2\pi\mu a^3 \alpha_{imm}. \end{aligned} \tag{75}$$

3. Discussion

In the derivation above, we assumed that the velocity of the fluid relative to the center of the sphere is small, which allowed us to neglect the nonlinear terms in the momentum equations. For sand particles suspended in real rivers, this assumption is acceptable. For example, the average diameter of suspending sand particles in the Yellow river is around 0.03 mm. The sedimentation velocity of a sand particle of such size in water due to gravity is very small. Moreover, the inertial effects of such fine particles are also small, thus the particles shall not fall much behind the water flow due to the inertial effects. Taken these two reasons together, the nonlinear terms can be neglected. On the other hand, for sand particles moving along the river bed, which are usually of diameters above 0.2 mm, their relative velocities with respect to the water are much larger, thus the assumption we made above is not applicable.

In the discussion, we also assumed that the length scale of the flow is much larger than the diameter of the sand particles. In rivers, the length scale of the flow is roughly of the same order of the water depth, which is typically a few centimeters to a few meters, indeed much larger than 0.03 mm.

In the derivation, we applied the initial condition $W_i = 0$, which, strictly speaking, is rather arbitrary. On the other hand, what we are interested in is not the exact flow field at early times, but the flow at larger t. For the viscous flow around the sphere, the influence of the initial condition disappears quickly as t increases. Therefore, from a practical viewpoint, the arbitrary choice of the initial condition $W_i = 0$ has negligible effects.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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